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THE DISCOVERY OF NON-EUCLIDEAN GEOMETRIES AND ITS CONSEQUENCES: OBSERVATIONS ON THE HISTORY OF CONSCIOUSNESS IN THE NINETEENTH CENTURY

Renatus Ziegler

1. Introduction

The discovery of non-Euclidean geometry is rightly considered as one of the most influential achievements of the nineteenth century. The structural changes in mathematics and mathematical physics caused by this discovery had important consequences for the development of the conception of the world held by scientists and philosophers. Even artists were influenced and inspired by it (see *Henderson*, 1983). In this article I would like to examine one aspect of this large set of different courses of development which intermingle and influence each other, an aspect which is especially important for the history of consciousness, namely, the significance of the discovery of non-Euclidean geometry for the development of the philosophical understanding of mathematical methods in the natural sciences.

Towards the end of the Enlightenment, those scientists who considered mathematics as the prime example of all science were the most highly esteemed. Their way of thinking mainly influenced the teaching staff at the Ecole Polytechnique in Paris and other similar institutions of higher learning. Mathematical method (in so far as it was considered at all) was understood not only as the *application* of mathematical *content,* but also as a general methodological principle of scientific cognition according to which the form of various scientific areas of study should be oriented following the rigorous structure of mathematics (especially that of Euclidean geometry). For various reasons, however, in the course of the nineteenth century, what became the dominant view was the opinion that with "mathematical methods" only the description of empirical conditions of facts by means of mathematical content (theoretical models) could be intended, and this was a view that became a determining factor for the further development of the natural sciences and has remained decisive up until the present day. In this article it will be shown to what extent and how the discovery of non-Euclidean geometry is responsible for this development. Careful examination of the origin of this decisive change in consciousness about the middle of the nineteenth century furthermore provides clear insight into the powerful consciousness-producing capabilities of mathematical thought which, if properly controlled and managed, can lead to overcoming the one-sided character of mathematical methods in the natural sciences and thus to the prospect of a renewed understanding of mathematical physics and of an organic mathematical view of the world.

2. *Euclid's* Elements *and the Consequences*

Without Euclid there can be no non-Euclidean geometry. Euclid was the first to recognise the special nature of the geometry named after him. Today we know that Euclid in his *Elements* (circa 300 B.C.) for the most part "only" summed up the results of his predecessors and contemporaries and put them into a textbook that proved to be the most successful mathematics book of all times and in the number of its editions as well as its cultural and geographical divulgation was second only to the Bible. Although one naturally as a mathematician in the 20th century has many objections to various details in the Euclidean system, the *Elements* were nonetheless for over two thousand years *the* prime example of mathematical rigour and clarity. What makes Euclid truly immortal is his really prophetic insight in the choice of the explicit assumptions of "his" science, geometry. He demonstrates (at least to a certain extent) that geometry is based on specific undemonstrable assumptions that can be seen to be so and which are called by him definitions and postulates (or axioms) and from them everything else can be deduced by purely logical deduction. Naturally Euclid's system of axioms is neither complete, independent or free from circular definitions. These points are simply small details if one considers that this was the first attempt of its type to develop such a system. Euclid's genius is apparent from the fact that he considers a particular geometrical statement as a postulate (or axiom) which is neither simple nor intuitively clear even though that is not what one usually expects from postulates (see figure 1):

Figure 1

Postulate 5: *If a straight line intersects two straight lines and forms interior angles on the same side, and these two angles are together smaller than two right angles, then these two straight lines, if they are lengthened to infinity, will eventually meet on the side on which the two angles are smaller than two right angles.*

It is not surprising that a large number of prominent mathematicians could not accept Euclid's assumption that this complicated statement has to be a postulate. A detailed analysis of the structure of the *Elements*, especially of the first Book, makes it clear that early commentators had doubts about the derivability (possibility of providing a proof) of this statement from the other postulates and theorems. The problem at first seems to be clear and elementary, but, in the course of centuries, turns out to be extremely complex and seemingly unsolveable (see for example *Bonola/Liebmann*, 1908; *Mainzer*, 1980; *Pont*, 1986).

Euclid calls two straight lines lying on the same plane *parallel* if they do not intersect at any point when they are lengthened to infinity on both sides. Those who had doubts about Euclid's system produced many statements which are equivalent to Euclid's Fifth Postulate, and one of them is the following, which is called here the Parallel Axiom PA(E) (see figure 2):

Figure 2

Parallel Axiom PA(E): *There is only one line parallel to a straight line g and passing through a point P, which is not on g.*

Only very few geometricians after Euclid had any doubts about the provability of PA(E), that is, about the assumption that PA(E) is a theorem and not a postulate. The question, however, was how could this assumption be checked? The "most successful" attempts consisted in the construction of an internally coherent system of geometrical theorems, which is based on a statement that contradicts PA(E), but otherwise is in agreement with all the other postulates and with theorems that are independent of PA(E). The whole idea of the undertaking is to derive from this system a contradiction to the other postulates (or to theorems that are dependent on PA(E)). The discovery of such a contradiction would be a sure sign of the provability of PA(E) (indirect proof method).

None of these critics, and some of them were extraordinarily clever, ever thought that there could be a "geometry" in which PA(E) was *not* true (that is, that PA(E) could be logically independent of the other postulates). After all, what was geometry if not *the* science of space? Right here, in our world of sense perceptions, the validity of PA(E) and the theorems that could be derived from it could not be put in doubt. One must make this inseparable connection of classical geometry with the empirical world of our senses and its corresponding space completely clear in order to understand fully the significance of subsequent developments. Until well into the nineteenth century geometry was identical with the conceptual description of empirical circumstances of the world. Geometry was not a model, it was the thing itself. Its strictly logical character did in fact differentiate it from other sciences but did not separate it from them in principle.

3. *The Discovery of a Hypothetical Non-Euclidean Geometry*

At the beginning of the nineteenth century Carl Friedrich Gauss (1777-1855), Nicolai Ivanovich Lobachevskij (1792-1855) and Janos Bolyai (1802-1860) came to the conclusion that PA(E) cannot be proven (see *Bonola/Liebmann*, 1908; *Reichardt*, 1976; *Mainzer*, 1980). What arguments could they provide for their thesis? At that time there was neither any empirical evidence in favour of the assertion nor a mathematical field in which a corresponding example (model) could be constructed. Like many of their predecessors, Gauss, Lobachevskij and Bolyai also produced a coherent system of theorems free of contradictions (mainly based on trigonometry); this system was clearly in contradiction with PA(E) but in complete harmony with all other axioms and derivable theorems. However, this system was inimical to any reasonable conception of space. Right from the beginning the basic assumption which is presupposed in place of PA(E) seems absurd (see figure 3);

Figure 3

Parallel Axiom PA(GLB): *There is more than one line parallel to a straight line g, and passing through a point P, which is not on g.*

The thought structure built up by Gauss, Lobachevskij and Bolyai, in which PA(E) is replaced by the axiom PA(GLB), will be here called the *non-Euclidean geometry GLB*. Argumentation in the conceptual system of this geometry is necessarily purely conceptual (axiomatic) and has to be presented without the aid of a concrete realisation (model), that is, without concrete representations or drawings. The reason for this restriction is that the latter can only be taken from Euclidean geometry, and this should absolutely not be a presupposition here. Drawings, in the case that they are needed at all, can only have a symbolic character (as in figure 3) and no real meaning. What had become since the eighteenth century common property of mathematicians in number theory and analysis, namely the separation of mathematical concepts from their reference in the world of the senses, that is, the understanding of its content in the form of pure thoughts or laws, is now for the first time accomplished in the field of geometry. In the case of geometry the change to thoughts that are free from sensory perception is particularly difficult because the content of geometry, in contrast to number theory, is from the beginning directly connected to the world of the senses, and what is evident at first sight is not sufficiently clearly differentiated from what can be logically deduced. This differentiation can be achieved only by means of a vigorous effort on the part of consciousness, which must reject the perceived portion of the notion of geometry and develop the portion that is purely conceptual. Establishing such a system which is absurd from an empirical point of view, as the non-Euclidean geometry GLB, requires extremely intensive thinking. How much faith in the power of pure thought there is in the assertion that this system, simply on the basis of its *logically* consistent structure, is just as valid as Euclid's system, which is confirmed by any and all experience based on the *senses*. This faith was a result of the clearly experienced truthfulness of the content of pure thought. Therefore I would like to call this point in time *the hour of birth of pure thought in geometry* (see *Unger*, 1940).

Only a few decades earlier I. Kant (1724-1804) had claimed that (Euclidean) space had to be considered as the *condition* of the possibility of the perception of the appearances and not as a characteristic that depends on them; space is a "(...) conception that is a priori, it is necessarily the basis of the outward appearances of things. The self-evident certainty of all geometrical postulates is based on this a priori necessity (...)" (*Kant*, 1781, A24). After the discovery of the non-Euclidean geometry GLB, however, along with the usual (Euclidean) *conception* of space, there was a second one that could not without further ado be made to agree with empirical data. Consequently, the problem of the determination of the specific laws governing the outward order of the objects of the world of the senses became one of cognition, and *Euclidean* space could no longer be generally considered as the a priori condition of the possibility of the perception of the appearances of things.

4. The Copernican and Non-Euclidean Revolutions: A Comparison

The discovery of the non-Euclidean geometry GLB and its consequences were continually compared with the Copernican revolution. In fact, this comparison is extremely interesting for the history of thought, and for this reason it should be examined in some detail. Neither the Copernican system, nor the non-Euclidean geometry GLB were based on any empirical evidence (available at the time they were developed). They were not scientific discoveries in the usual sense, but rather general theories that derived their validity solely from their clarity, explicitness and logical consistency. They were symptomatic of the progressive evolution of consciousness and not the result of any sense observations or experiments.

One may object that the Copernican system is simpler and can be understood more easily than the Ptolemaic system that preceded it. In addition, it permits the calculation of much more accurate astronomical tables. But if one examines carefully the original system of Copernicus, which is to be found in his major work *The Movements of the Heavenly Bodies (De revolutionibus orbium coelestium)* (1543), one discovers that the details of his system are at least as complicated and artificial as those in Ptolemy's *Almagest* (circa 150 A.D.). This is based on the fact that Copernicus also allows only regular circular movements to trace the orbits of the planets and, therefore, his system is kinematically equivalent to Ptolemy's. Consequently, the astronomical tables that are calculated on the basis of his system are not at all more reliable than those going back to the traditional Ptolemaic system. The situation really improves only after the introduction of elliptic orbits for the planets, which was due to Kepler.

The Copernican system had a decisive advantage over the non-Euclidean geometry GLB: there was no contradiction with the world of the senses. This difference underlines the specific contrast between the evolution of consciousness in the 15th and 16th centuries on the one hand and the evolution of consciousness in the 19th and 20th centuries on the other hand, which was basic to the discovery of these systems of concepts. In the Renaissance scientific consciousness freed itself from the dogma of religion and from degenerated scholastic philosophy, it turned its attention to the experiential world of the senses and thus became conscious of itself and of nature as it appeared to man. In the 19th and 20th centuries the consciousness of the natural sciences overcame the dogma of the world of the senses and turned its attention to the observation of thought in order to learn about and understand fully the essential being of thought and the effective laws of nature. The latter development has at the present time in no way come to an end, it has only just begun. It is not, therefore, futile and "only" of historical interest to examine in what follows the role of mathematics, and especially that of the discovery of the non-Euclidean geometry GLB in this "adventure of reason".

5. From Romantic Natural Philosophy to Modern Scientific Method

In geometry the liberation of consciousness from the restrictions of a view of the world based on the senses was a step that had already been taken in philosophy by German idealism, thanks to Johann Gottlieb Fichte (1762-1814), Georg Wilhelm Friedrich Hegel (1770-1831) and Friedrich Wilhelm Joseph Schelling (1775-1854). For example, Hegel, in *The Science of Logic (Wissenschaft der Logik)* , published in 1831, starting from being, tries to derive the basic categories of the world of regular laws with the help of the dialectical self-determination of concepts and, by so doing, to make explicit the implicit determination of concepts. Here mathematics naturally can have only a secondary role, as Hegel makes clear in the forward to the first edition of *The Science of Logic* :

"Philosophy, if it is to be a science, cannot (...) in this case borrow its methods from a subordinate science, like mathematics, just as it cannot simply be content with categorical assertions of personal opinion or resort to reasoning based on superficial reflection. But it can only be the *nature of the content*, which is *active* in scientific knowledge, in that it is at the same time the *selfsame reflection* of the content, *which* first fixes and *produces its own determination.*" (*Hegel*, 1831, Forward to the first edition, p. 16)

Hegel clearly understands that he cannot deal with *active* laws of nature using the means that philosophy furnishes. He can only draw an inanimate outline of these laws as it appears in human consciousness by means of pure thought. On the other hand, he makes it clear that philosophy is useful for training pure thought:

"The systematic structure of logic is a realm of shadows, a world of simple beings freed from all shaping by the senses. The study of this branch of knowledge, staying and working in this world of shadows, is the complete formation and cultivation of consciousness. Its business is far removed from a view of the world based on the senses and goals in that world." (*Hegel*, 1831, Forward to the second edition, p. 55)

Logic is the place where the ideal world of pure laws can be seen in its purest state: "Logic is accordingly to be understood as the system of pure reason, as the realm of pure thought. *This realm is the truth as it is in and for itself without a veil.* One can therefore say that this content is *the representation of God as he is in his eternal being before the creation of nature and of a mortal soul.*" (*Hegel*,1831, Forward to the second edition, p. 44)

It is only natural that Hegel also wanted to make his point of view valid in the philosophical treatment of the natural sciences. He did not at all want to ignore the solid results of contemporary natural science and dedicate himself to vacuous speculation about how nature *should* be. The object of natural philosophy is the same as that of natural science itself:

"In so far as natural philosophy is *conceptual* contemplation, it has the same *universal* as its object *for itself* and considers it in its *own, immanent necessity* in accordance with the self-determination of the concept.

In the general introduction we already spoke about the relationship between philosophy and empiricism. Philosophy must not only be in agreement with the experiences drawn from nature, but the *origin* and *formation* of philosophical knowledge must also have empirical physics as a presupposition and necessary condition. But the process of origin and the preparatory work of a branch of knowledge is one thing, the branch of knowledge itself is another. In the latter can those things no longer appear as a basic foundation, which here rather has to be the necessity of the concept:" (*Hegel*, 1830, §246, p. 15)

Hegel's natural philosophy and other philosophical systems having nearly the same import only temporarily influenced the development of the natural sciences, which were changing their aspect at the beginning of the nineteenth century. New tendencies had already been noted at the end of the eighteenth century; they were destined to get the upper hand and, consequently, decisively influence the further development of the natural sciences. One of these tendencies was neo-humanism, and the other was the rationalism of the French Enlightenment and its culmination in the French Revolution. A researcher of great repute, Alexander von Humboldt (1769-1859) proved to be the most important figure, at least in Germany, during this period of transition. On the one hand, because of his background and the connection with his brother Wilhelm (1767-1835), he was interested in neo-humanism. On the other hand, because he was well-travelled and, above all, thanks to his stay in Paris from 1807 to 1827, he was dedicated to the ideals of the French Enlightenment. He was, therefore, practically made-to-order for achieving a synthesis of these two tendencies, which had been proceeding more or less independently of each other. From neo-humanism he adopted the ideal of pure knowledge as a goal in itself while neglecting its practical application, and from the rationalism of the French Enlightenment he adopted the analytical, rational and objective observation of nature. Because of this synthesis of these two powerful spiritual tendencies of the eighteenth century, Humboldt proved to be the spiritual father of the character of the natural sciences at the end of the nineteenth century.

In Paris Alexander von Humboldt learned the mathematics that was flourishing there at that time and the mathematical physics that was just getting started. He came to respect these subjects highly. The new methods and structures adopted in education at the university level during the French Revolution produced impressive and very solid results in the natural sciences and engineering thanks to a rigorously applied general orientation that was based on mathematics. The conclusions that Humboldt was practically forced to draw from his observations of what was going on in Paris brought him necessarily into a sort of conflict with the idealistic natural philosophy that was flourishing at the time in Germany. The reason for this was that mathematics, as mentioned above, had only a subordinate role in German natural philosophy.

Alexander von Humboldt's transfer from Paris to Berlin in 1827 marked the beginning of a new phase in the reorganisation of all branches of knowledge in Prussia, especially in the natural sciences. If, at the beginning of the 1820s he had already made his influence felt on the situation in Germany, he was able to be even more influential after 1827 because of his being close to the king and the education department. "He possessed (...) the rare gift, even without a full understanding, of grasping the significance of things in areas of knowledge that were entirely foreign to him. In fact, thanks to his ability to understand things and a certain feeling for what was needed at the time, he was able to suggest fruitful applications of branches of knowledge that he had not studied. Closely connected to this characteristic was his almost instinctual ability to spot young promising people who had talent before they had even done anything. Furthermore, since Humboldt enjoyed in Berlin an extraordinary social position, which made him extremely influential because of his relations with the Court and his numerous personal connections, in Prussia he was the right man to determine for years on end the development of the sciences that interested him. In this way he can be considered to be the person responsible for a phenomenon in the area of mathematics and the natural sciences which had begun about ten years earlier for the liberal arts, namely in 1810 with the foundation of the University of Berlin (by his brother Wilhelm), and which I would like to call the 'German scientific Renaissance'." (*Klein*, 1926, p.17)

How much influence A. von Humboldt had on new ways of thinking about the natural sciences in general can perhaps best be seen from the decisive change that resulted from his being chairman of the seventh assembly of the "Gesellschaft Deutscher Naturforscher und Ärzte" (Society of German Natural Science Researchers and Medical Doctors) in 1828 in Berlin, a society that subsequently became very influential (see *Degen* , 1956). Originally bound up with the natural philosophical ideas of German idealism due to the connection with its founder, Lorenz Oken (1779-1851), Humboldt brought about in 1828 a basic change of emphasis in favour of the spirit of modern natural science. Thanks to the *Kosmos* lectures that were already at that time famous, given in the winter semester of 1827/28, the field of knowledge was clearly defined: there was to be a clear change in direction away from the idealistic natural philosophy that was very common in Germany to an unprejudiced, methodically and technically perfected description of nature.

Although Alexander von Humboldt did not conceive of any particular one-sided method, the natural sciences subsequently almost exclusively used mathematical concepts for precise descriptions of natural phenomena. Once that faith in idealistic philosophy, which had been elevated to a science by Hegel and others, had been lost, it was thought that the only way to be able to reach a reasonable level of certainty in the formation of concepts in the natural sciences was to apply the clarity that one had experienced in mathematics. To support this line of argument, it was possible to rely on the authority of Kant, who had already in 1786 determined: "But I claim that in the study of any branch of nature one will find only so much *real* scientific knowledge as there is mathematics there." (*Kant*, 1786, Forward, A VIII)

But Johann Wolfgang Goethe (1749-1832) had already in 1793 noted that it was not simply a question of applying mathematics to the natural sciences. It was rather practising the mathematical *method*, namely, carefully and continually reducing complex phenomena to simple basic facts which cannot be further broken down, things that he called "basic phenomena":

"This circumspection, that is, putting things in the right order or rather ordering what follows from what goes before and then putting these things in their correct logical order, is something that we have to learn from the mathematicians and which we have to apply even where we do not use any calculations, just as if we had to give an exact account of what we have been doing to a most exacting geometrician. It is simply a question of what the mathematical method is: it makes every step immediately clear in the assertion because of its circumspection and purity, and mathematical proofs are really only detailed comments explaining that that which has been brought under consideration was already there in its most simple parts and in the whole sequence, was seen in its entire extent and under all conditions was found out correctly and without any objection." (*Goethe*, 1793, p. 19)

Giving exclusive preference to mathematical concepts in description and analysis of natural phenomena is based accordingly on a confusion of the mathematical method*, which provides true certainty, with the* contents *of mathematics*. This confusion is no historical accident, but it is a symptom of the fact that most scientists of the second half of the nineteenth century felt sure about what they were thinking only when they were working with mathematical concepts. The definitive separation of geometry from sense perception, which was brought about by the integration of the conceptual system of the non-Euclidean geometry GLB in the whole field of mathematics, completely liberated mathematical modes of thinking from the restrictions of sense perception and consequently elevated mathematics to a new height of absolute purity and (relative) perfection based on nothing else than itself. This change necessarily had consequences on the contemporary process of mathematisation of the natural sciences. But this development also had profound significance for the development of consciousness in general because it made quite clear how mathematical modes of thinking strongly influence the growth of consciousness. We now want to consider this development, but we must first examine some details of the history of mathematics.

6. The Reception of Non-Euclidean Geometry

The first works by Lobachevskij and Bolyai on non-Euclidean geometry were published in the third decade of the nineteenth century. The history of their acceptance does not really start until the end of the sixties. Some of the reasons for this late acceptance and divulgation were already mentioned above. The events which began this development (see *Bonola/Liebmann*, 1908; *Reichardt*, 1976) throw light on the obstacles that had to be overcome.

The acceptance of the conceptual system of the non-Euclidean geometry GLB was hindered by the identification of Euclidean geometry with the scientific conception of space. Kant misplaced the laws of geometry, considering them to be an essential part of our means of conception and consequently concluded that geometry was an apodictic science, that is, that its laws were valid a priori and independent from experience. Hermann von Helmholtz (1821-1894), one of the founders of empiricism, was the first to express serious doubts about the a priori certainty of (Euclidean) geometry. He was of the opinion that the specific character of space was a question of experience and therefore could be approached by means of empirical research (see *Jammer*, 1980). His article, published in 1868, *The Basic Facts of Geometry* (*Ueber die Tatsachen welche der Geometrie zu Grunde liegen*) appeared at about the same time as the famous inaugural lecture of Bernhard Riemann (1826-1866) *The Basic Hypotheses of Geometry* (*Ueber die Hypothesen welche der Geometrie zu Grunde liegen*), given in 1854, posthumously published in 1867). Even though, from a mathematical point of view, the results of Helmholtz and Riemann are closely related, the titles of their works already reflect an important difference: Helmholtz researched the physiological and empirical basis of geometry and its consequences for the mathematical structure of space while Riemann speculated on the structure of physical space on the basis of purely mathematical criteria.

The reception of the non-Euclidean geometry GLB received an important impulse from another quarter through the publication of the correspondence between Gauss and the astronomer Hans Christian Schumacher (1780-1855) in the years 1860-1863. This was the first opportunity the general public had to learn more about the extensive though hitherto unpublished research of Gauss on the problem of parallels in which he had already some years earlier come to the same conclusions as Bolyai and Lobachevskij (see *Reichardt*, 1976). The authority of the "prince of mathematicians" (Princeps Mathematicorum), as Gauss was called during his lifetime, contributed significantly towards furthering the dissemination of the writings of the two hitherto practically unknown geometricians and to expedite translations of their work into German, French and Italian.

It is an interesting fact in the history of the development of consciousness that the "rediscovery" and dissemination of the non-Euclidean geometry GLB at the end of the sixties and beginning of the seventies of the nineteenth century proceeded in close connection with the subsequent further development of projective space in more than three dimensions, in respect both to personal contacts and the content of work done (see *Ziegler*, 1985, p. 122ff.). Hermann Grassmann (1804-1877) presented the first systematic treatise of a geometrical calculation for affine spaces of any number of dimensions with his *Studies in Extension* (*Ausdehnungslehre*, 1844). This study, however, was given hardly any attention by established mathematicians. Julius Plücker (1801-1868) succeeded in making the first real breakthrough into the fourth dimension with a work on line geometry that was outlined in 1846 and then further developed from 1865-1868, and which was given the title *The New Geometry of Space, Based on Considerations of the Straight Line as an Element of Space* (*Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement*, 1868/69). However, he himself rejected research into spaces of any number of dimensions as "metaphysical". Arthur Cayley (1821-1895), on the other hand, had no hesitation whatsoever in introducing as early as 1843 n-dimensional projective spaces of a purely algebraical sort. In 1870 he provided a systematic and further developed explanation of his earlier work.

Riemann, in his inaugural lecture, went so far as to restrict his discussion of the question exclusively to the general case of an n-dimensional space, and he gave no concrete examples at all. Consequently, his lecture was quite abstract and almost incomprehensible. The immediate influence of his thinking on mathematical physicists, whom he was really addressing, was, therefore, very little. It was only in the twentieth century that his considerations on the structure of physical space were taken up again, mainly thanks to Albert Einstein (1879-1955). But the influence that the publication of Riemann's inaugural lecture had on the development of mathematics can hardly be overestimated. As the non-Euclidean geometry GLB gradually became known, mathematicians realised that Riemann had outlined a research programme that fused together this hypothetical GLB geometry, hypothetical n-dimensional geometry and Gauss's differential geometry. This programme bound them up and produced a higher-level unity, thereby opening up previously unknown possibilities for research in mathematics. Riemann suggested that the non-Euclidean geometry GLB could be realised within the limits of a special case of his concept of an "infinitely extended manifold of variable metric" (and consequently variable curvature), namely, within a "manifold of constant *negative* curvature". Riemann further developed, that there exists besides the non-Euclidean geometry GLB another hypothetical geometry, which has to do with his „manifolds of constant positive curvature“ and for which were valid neither PA(E) nor PA(GLB), namely the postulate set by Gauss, Lobachevskij and Bolyai. Summing up, one can say that Riemann sketched out a comprehensive *analytical* theory of hypothetical geometries in contrast to the purely *axiomatic* (conceptual) procedure of the discoverers of the non-Euclidean geometry GLB.

Shortly after the publication of Riemann's lecture, the Italian mathematician Eugenio Beltrami (1835-1900) verified that, in fact, the non-Euclidean geometry GLB can be realised by means of Gauss's differential geometry when applied to surfaces of constant negative curvature (considered as "two-dimensional manifolds of constant curvature" in the sense given to it by Riemann) (see *Bonola/Liebmann*, 1908; *Mainzer*, 1980). That means that the geometry *on* one of such curved surfaces obeys exactly the laws that Gauss, Lobachevskij and Bolyai had found on the basis of purely axiomatic considerations. In other words, Beltrami constructed the first concrete mathematical model of the non-Euclidean geometry GLB. This was an important step for the recognition of non-Euclidean geometry: it was no longer an isolated area of mathematical-logical speculation, but firmly established as part of the most up-to-date results of the mathematical research of the time, and thus elevated from the low rank of a speculative conceptual system to the status of a mathematical fact (theory).

7. Non-Euclidean Geometry and Projective Geometry

Riemann's initial step towards the integration of the non-Euclidean geometry GLB in contemporary mathematics mainly benefited development in the area of analysis. The traditional area of geometrical research in the nineteenth century, projective geometry, remained entirely untouched for the time being.

Just like non-Euclidean geometry, projective geometry was born at the beginning of the nineteenth century (see *Klein*, 1926; *Mainzer*, 1980). It was mainly Jean-Victor Poncelet (1788-1867), who took up the ideas of his predecessors in a most original way, that was responsible for the breakthrough made by projective geometry, due to his "new manner of geometrical perception" and his "projective thinking" (*Klein*, 1926, p. 80). The plan of a *pure synthetic* construction of projective geometry can first be found in the works of Jacob Steiner (1796-1863). He developed the principle of the successive generation of figures of a higher level from linear figures (like ranges of points, pencils of lines, etc.), as, for example, the generation of the points of a conic section, or simply conic, from two projective pencils of lines. Once that he had prepared a plan, he kept on working at it diligently, and it was to this project that he applied his splendid gifts of lively teaching. Synthetic projective geometry reached a certain conclusion and culmination due to two thoroughly systematic constructed works which were organized according to the principle of duality: *Geometry of Position* (*Geometrie der Lage*, 1847) and *Contributions to Geometry of Position* (*Beiträge zur Geometrie der Lage*, 1856-60) by Karl Georg Christian von Staudt (1798-1867). The real founder of *analytical* projective geometry is Julius Plücker (1801-1868). He published not only a pair or articles or *one* book on this subject but dedicated his entire life's work to this area and accordingly exercised great influence on its development. Pluecker was less interested in the production of new results than in the demonstration of new *methods*: he wanted to provide a new structure for analytic geometry. Some of the things that he was responsible for are the consistent use of homogeneous coordinates, the analytical formulation of the principle of duality and the principle of abbreviated notation.

In a work on algebra dating from 1859, which has become classic, Arthur Cayley posited the typical characteristics that distinguish Euclidean geometry from projective geometry. He started out from the facts that had been known since Poncelet's time, namely, that all Euclidean transformations of the plane, considered from a projective point of view, leave invariant the line at infinity as well as a (conjugate complex) pair of points lying on it. He called this pair of points the "absolute" pair of points of Euclidean geometry. Cayley realised that these Euclidean transformations were special cases of much more general transformations, namely, projective transformations which leave a conic invariant, that is, have a conic as "absolute" figure. He was able to show that it is possible, on the basis of such a system of transformations, which leave a conic invariant, to develop a "projective *metric* geometry". This geometry is clearly different from Euclidean geometry in many respects (for this and what follows, see *Bonola/Liebmann*, 1908; *Klein*, 1926, 1928; *Mainzer*, 1980).

Figure 4

It was Felix Klein (1849-1925), who forged the connection between Cayley's "projective metric geometry" and the non-Euclidean geometry GLB. He was fully up-to-date with the most recent developments in all aspects of geometrical research and was barely twenty-two years old. If one takes as absolute figure a real conic, which is not degenerate, and considers only the area with points "on the inside", it is possible to realise the hypothetical non-Euclidean geometry GLB with this set of points on the condition that one interprets the points on the conic as "infinitely distant" and identifies the movements (isometries) of this geometry with those projective transformations which leave this conic invariant. For every straight line *g* and for every point *P* which does not lie on this line, there are exactly two parallel lines *u* and *v* and an infinite number of non-intersecting straight lines (figure 4), that is, PA(GLB) is satisfied. Furthermore, with the application of an appropriate definition of the length of a line section *AB* by means of the logarithm of the (projective) cross-ratio of the points *A*, *B*, *U*, and *V*, all the postulates of Euclidean geometry, with the exception, naturally, of PA(E), are satisfied. This is consequently a (projective) model of the non-Euclidean geometry GLB. Klein called this geometry "hyperbolic geometry", and he called the geometry which has an imaginary conic as its „absolute“ or basis "elliptic geometry".

Klein also showed that this "elliptic geometry" is isomorphic to the geometry first mentioned by Riemann for two-dimensional manifolds of constant positive curvature. This is, however, really close to the geometry on the surface of a sphere, that is, classic spherical geometry. If one calls the circles (the largest circles) on the sphere's surface "straight lines" and pairs of diametrically opposed points "points", then neither PA(E) nor PA(GLB) are satisfied by this spherical geometry that has been modified and changed in this way into elliptic geometry. Instead of these parallel axioms there is another one that is valid (see *figure 5*):

Figure 5

Parallel Axiom PA(R): *For a straight line g and a point P not liying on g there is no parallel line.*

In other words, in elliptic geometry there are no straight lies that do not intersect (since pairs of circles intersect in a pair of diametrically opposed points). This fact places elliptic geometry closer to projective geometry and, consequently, for elliptic geometry the axioms of Euclidean geometry, apart from PA(E), are only partially applicable (especially the Euclidean axioms of order are not applicable).

Klein's idea of inserting into projective geometry the non-Euclidean geometry GLB and Riemann's geometry in a two-dimensional manifold of constant positive curvature (sphere) was, in fact, the central point of his famous *Erlangen Programme* (1872). This research programme, even if its content was completely different, had a wide-reaching influence on the development of mathematics similar to that of Riemann's. Klein employed the idea of a "group", which had only recently entered mathematical consciousness, as the organising principle of an all-embracing definition of the concept "geometry". In his view, the task of the geometrician consists of establishing different types of groups of transformations and then working out for each type those characteristics and relationships which do not change, that is, remain invariant with respect to their structure. In other words, geometry is the invariant theory of a group of transformations. In this way, the projective transformations in the plane that leave a conic invariant can be divided into different types, and the type depends on whether the conic is real, imaginary or degenerate with two conjugate complex points or with two conjugate complex lines. All geometrical theorems with which these transformations are respectively in agreement make up the content of "hyperbolic", "elliptic", "Euclidean" or "polar-Euclidean" geometry. Thus, for example, the above-mentioned axiom PA(GLB) of "hyperbolic" geometry is apparently valid and takes the place of PA(E) whenever the planes are transformed under the conditions that these transformations bring straight lines into straight lines and leave the absolute (real) conic invariant. The same is true for the invariance of PA(R) in "elliptic" geometry on the surface of the sphere if the transformations are identified with the (Euclidean) rotations of the sphere.

Klein's ideas proved to be extremely fruitful for geometrical research for a long time even in the twentieth century. They were the triumph of a first-class synthesising mind and helped in achieving a breakthrough for the recognition of non-Euclidean geometry. Klein's ideas showed geometricians the place where they could find within the limits of projective geometry, which was at its highest point, a realisation of the axiomatic system of Gauss, Lobachevskij and Bolyai as well as that of Riemann's manifolds of constant curvature.

The discovery and further development of non-Euclidean geometries and their models led to "non-Euclidean spaces" that were not identical with space as we usually consider it but which were conceivable in a precise manner. As already noted, because of this, the restrictions, which the three-dimensional quality of our way of usually considering space imposed, began to be loosened: one now took n-dimensional point-spaces just as seriously *on a formal level* as the already well-known "intuitive" spaces of a higher number of dimensions as, for example, four-dimensional line space. Once the consciousness of mathematicians had been freed in *one* place from the Euclidean way of looking at things, there were no longer any limits to what could be possibly thought or understood.

The developments that have been outlined so far led to a revolution in consciousness that started in mathematics in the seventies of the nineteenth century and spread to the entire culture, and we today are still experiencing its consequences. Non-Euclidean geometry was *one* cause of it. Somewhat later two other streams of development were added, and they were decisive for everything that followed. The first was the foundation of set theory by Georg Cantor (1845-1918). This shed light on the chasm of the concept of infinity and brought conceptual clarity to areas in mathematics which are entirely foreign to perception or intuition. The second was the origin of the axiomatic method, and a third was the elaboration of mathematical logic as a separate discipline within the basic research into the foundations of mathematics.

Confidence in the inner consistency of mathematics and its totally unlimited possibilities grew at an unbelievable rate. Mathematical concepts had been freed from the fetters of mental representation and perception, they had been clearly ordered and arranged in respect to their own inner relationships. Now they were finally elevated to the ideal of all serious endeavours in the natural sciences in general. It was entirely in keeping with the views of Kant that one believed that it was possible to have conceptual clarity only where there was mathematics, that is, where mathematical concepts were employed for analysis and synthesis of processes and patterns. Mathematics was not only *an* instrument for gaining knowledge in the natural sciences, but it was *the* conceptual network in which the total multiplicity of nature was ultimately to be apprehended. The mathematization of the natural sciences thus entered its decisive phase.

This development had as a result that also the non-Euclidean geometries were examined to determine their potential value in making models of selected areas of nature. No empirically verifiable success was obtained for the time being. But the purely theoretical application of non-Euclidean concepts to the mechanics of rigid bodies gave results which became critical points for a wider conception of mathematical physics. These developments, along with some other topics, will be the subject of what follows in the next sections.

8. Applications of Non-Euclidean Geometries in Mechanics

It is an interesting historical fact that both of the mathematicians who were mainly responsible for the integration of the non-Euclidean geometry GLB into the mainstream of mathematics were also trying at the same time to find possible applications of this new mode of geometry in mathematical physics. We have already referred to the suggestions of Riemann in this respect. However, up to the present time, it has hardly come into mathematical or, for that matter, historical consciousness that non-Euclidean geometry, in the form in which Cayley and Klein realised it in projective geometry, was applied in a detailed manner to a classic area of physics, namely, to the mechanics of rigid bodies.

Felix Klein generalised the ideas of Cayley and applied them to three-dimensional and higher-dimensional spaces and demonstrated that in each of these spaces there are several fundamentally different types of non-Euclidean geometries (see *Klein*, 1928). Because of his teacher Julius Plücker, who was interested in both questions of line geometry as well as mechanics, Klein dedicated himself to integrating these areas into the system of his Erlangen programme and into projective geometry.

It is certain that Euclidean space has (in Cayley's sense) an degenerate projective metric and, consequently, in comparison with general projective metric geometry, is characterised by a significant lack of symmetry. This means that in the area of Euclidean geometry the principle of duality of projective geometry is no longer universally valid. In accordance with this principle, for every figure constructed by means of points (planes) there exists a similar figure with a corresponding structure but made up of planes (points). However, in Euclidean geometry there is only *one* isolated infinite plane but no *isolated* infinitely distant point. This is precisely a characteristic of Euclidean geometry since its absolute figure (in Cayley's sense) consists of this infinite plane and an imaginary conic lying in it. This figure, however, is not polar in respect to itself. The reason for this is that the dual of a conic is a cone. But in Euclidean geometry there is no such „absolute" cone that is invariant for *all* congruent transformations. However, there is such a cone in polar-Euclidean geometry; but this geometry has no absolute invariant conic. It is only by uniting both of these spaces that the asymmetry can be eliminated.

This asymmetry of Euclidean space can be seen particularly clearly in kinematics and statics. For example, rotations have clearly defined axes but translations do not. As August Ferdinand Moebius (1790-1868) first clearly saw, the basic concepts of statics are formally analogous to the basic concepts of infinitesimal kinematics. A single force has a well-defined axis while torque does not. On the other hand, according to Louis Poinsot (1777-1859), a pair of forces can be assigned a torque, and a pair of infinitesimal rotations can similarly be assigned a translation. Felix Klein supposed that these asymmetries disappear if one replaces the Euclidean metric with one that is not degenerate in Cayley's sense, that is, if one replaces the conic lying in the infinite plane with a surface of the second degree (quadric), which is not degenerate. In a space metricised in this way both rotations and translations must have well-defined axes. (For details of this theory and its historical development, see *Ziegler*, 1985)

A student of Felix Klein, Ferdinand Lindemann (1852-1939), developed in his dissertation (1874) the basic concepts of statics and kinematics in connection with non-degenerate projective metrics, that is, within the projective models of both of the above-mentioned classic non-Euclidean geometries, hyperbolic geometry and elliptic geometry. As in Euclidean statics, one then has to do with forces which work "by" straight lines (and are called translation forces); but now, in the place of torque, there is a force which works "around" a straight line (and is called a rotation force), and it has a working *plane* (dual correspondence to the working *point* of the translation force). Every movement is momentarily a screw-movement, but, in contrast to Euclidean movements, it has *two* well-defined axes. (A faint image of this general principle can be traced in connection with Euclidean geometry: the infinitely long straight line at a right-angle to the screw axis can be interpreted as the second axis of the screw.) Every such screw movement can be broken down into either two rotations (as in the case of Euclidean geometry) or into two translations with well-defined axes. In this case one axis can be freely chosen at any time, but then the second axis is clearly determined. (For a detailed historical and systematic analysis of Lindemann's dissertation as well as its prehistory and reception, see *Ziegler*, 1985.)

Lindemann gave great importance to the fact that his considerations had no connection with reality and thus took his distance from the metaphysical speculation that was going on at the time in respect to four-dimensional and non-Euclidean spaces. Forces of rotation, that is, torques with plane-like character, cannot, in fact, be empirically determined within the limitations of Euclidean mechanics. There are also no experiments in the entire area of the natural sciences that indicate the existence of such forces. Despite this, non-Euclidean mechanics of rigid bodies was pursued for a certain time and further developed in a detailed manner. Whether or not there are possible applications in the real world for which these concepts, which are not applicable in Euclidean space, can have any importance must remain an open question for the moment. It is an essential trait of mathematics that it can grasp the content of general principles with its concepts even though no evidence for the existence of these laws has been discovered or no perceptible phenomena at all can perhaps even be assigned to these laws.

The case of non-Euclidean mechanics is another instance of a creation in the realm of thought that is justified only by its internal logical consistency. At the same time it is an expression of the need for mathematicians to go beyond the limits set for them in the past and to address themselves to the general principles of subjects which seemed to be closed off a short time before because of the negating effect determined by the way of looking at things dictated by sense perception. This was an important step forward in overcoming the restrictions imposed by the experience of the senses in the area of mathematical physics and, at the same time, the beginning of an enlarged version of this discipline.

9. From the Natural Sciences to Spiritual Science

Towards the end of the nineteenth and the beginning of the twentieth century two intellectual tendencies became well-defined. They had been noticed continually since the Copernican revolution, but it was only in the nineteenth century that they made a breakthrough and gained ascendancy. On the one hand, we find in the history of the development of mathematics a tendency that became especially apparent in the nineteenth century, namely, the formation of concepts that were more and more abstract and free from sense perception. Their only justification was in their reciprocal relationships that functioned like general principles. This was the reason why mathematics became a branch of knowledge with purely intellectual or spiritual content. On the other hand, the natural sciences in the nineteenth century finally freed themselves completely from being included in one way or another in a cosmos whose basic matter was considered to be essentially of a spiritual nature. What interested scientists was only the formal aspect of natural laws and no longer the quality of the forces involved. The form of knowledge applied to inorganic nature was elevated to the form of knowledge in general. The separation inherent there between concept and object (*Steiner*, 1886, p. 105) corresponded in an ideal way to the predominant dualistic conception of the world. Furthermore, the interpretation of formality was limited to what is mathematical, and the predominant scientific method became the mathematical model. The "new spirituality" of mathematics, that is, its total abstractness, general quality and flexibility made it an apparently ideal and universal instrument for the acquisition of knowledge, and attention was no longer given to its main limitation. Instead of employing the help of mathematics to make man conscious of the essential character of the spiritual and its relation to the world of the senses, mathematics was employed as a simple instrument for dominating nature.

In what follows the ideas of Rudolf Steiner (1861-1925) about the importance of mathematics for the development of supersensual (transcendental) consciousness as well as for the scientific method will be considered. The long quotations serve to show that Steiner makes it absolutely clear what is being discussed.

Rudolf Steiner was fully immersed in the Platonic tradition and also followed Goethe in his relation to mathematics. Steiner clearly expounds the importance of mathematics for the development of modern consciousness: "Mathematical intuition" is, as Steiner explains, "a means of learning how to live in the supersensual world of ideas. Mathematical figures are on the border between the world of the senses and the purely spiritual world. (...) By means of mathematical figures I can learn about supersensual facts as if they were sense perceptions. (...) The idea must be contemplated in a purely spiritual way, if it should be understood in its true essential nature. One can train oneself to do this if one practices the elementary stages in mathematics, if one clearly understands what one really gains by working with a mathematical figure." (*Steiner*, 1904, p. 84f.)

"*One* of the ways that lead to transcend the life in material nature is through training in the spirit of mathematics." (*Steiner*, 1904, p. 18)

Training of consciousness with the help of mathematics can be considered in two different ways. First, doing mathematics prepares a person for activity in the realm of *pure* general principles without the loss of consciousness and also prepares him for forming a concept of how pure laws are connected with each other and with sense perceptions. Second, regular practice with mathematical concepts improves one's *ability to think* and leads to the basic formation of a supersensual organ. We are now going to consider the first point: "Think of a 'circle'. In doing that, one does not think of this or that perceptible circle, a circle which perhaps has been drawn on a piece of paper, but of any circle at all that could be drawn or could be found in nature. The case is the same with all mathematical figures. They have a relationship with what can be perceived, but perception does not subsume their essence. There is an infinite number of possible forms of mathematical figures. When I think in a mathematical way, I think over and beyond what is perceptible and, at the same time, I am thinking in the world of perceptible things. The circle that can be perceived does not teach me the laws of the circle, but the ideal circle does, and it is alive only in my spirit. The perceptible circle is only a picture of the ideal circle, which could teach me about any and every other perceptible picture of the circle. This is the essential point of the mathematical way of looking at things (intuition). A single perceptible figure leads me out beyond itself. It can only be for me an image of a wide-reaching spiritual reality." (*Steiner*, 1904, p. 8f.)

"Learn to think about the essence of nature and of spiritual life in a manner that is free from the influence of sense perception to the same extent that the thinking of a mathematician is when he thinks about the circle and its laws. Then you can become a student of the occult. This should be written in large golden letters and be continually before the eyes of anyone who really seeks the truth." (*Steiner*, 1904, p. 15f.)

Summing up: "Knowledge that is gained in areas where the crutch of sense perception cannot be used is most easily understood where man is most free of such restrictions. This is so in mathematics, which is accordingly the easiest preparatory stage to get by for *the* occultist who wishes to elevate himself to higher-level worlds in bright luminous clarity and not in dark intuitive ecstasy or in dreamy intimations. The occultist and the mystic live in the supersensual in brilliant clarity just like the geometrician with his laws about triangles and circles. The true mystic lives in light and not in darkness." (*Steiner*, 1904, p. 15f.)

One should note that Steiner does not designate mathematics as the only possible and reasonable preparatory school for the acquisition of spiritual knowledge. It is only the easiest and most accessible. After what has been mentioned above, it should be clear that the study of Hegelian logic has a similar effect.

In principle, it is not important what the mathematical contents are for this kind of spiritual path. However, the more tenuous the connection is between the contents and the material world, the easier it will be to free oneself from that world. It is precisely here that there is an important aspect of the pursuit of projective geometry and of the non-Euclidean geometries that are embedded in it. But there is yet another point of view to be considered. In working with these geometries we learn not only to be able to deal with pure general principles, but we have to think about concepts for whose specification in concrete notions we require special preparation because no sensory perceptions are accessible to us that can provide such a specification. Consequently, this activity refines not only thinking but also the imaginative mental life that is still, for the time being, bound to the perceptible physical world. The former develops the capacity for pure thought, that is, thinking that no longer requires the support of sensory perception; the latter develops the capacity for thinking in a way that does not depend on sensory perception but which can still produce detailed imaginative creative thought within the limits of the usual mental life. The creative activity by means of which real notions are individuated from mathematical concepts without the help of modes of thinking determined by sensory perception corresponds exactly to the creative moral imagination of a morally responsible free human being who individuates a general precept of behaviour, understood by means of moral intuition, into actual instructions directing his or her behaviour (see *Steiner*, 1894, Chapter XII). Since the behaviour to be realised is in the future, the moral imagination can be supported by no present or past perception. Because the accessible sensory world has Euclidean character, we cannot rely on any present experiential perceptions in the course of the individuation of the laws of projective or non-Euclidean geometry into actual figurative mental representations. The production of such representations in one's inner way of seeing things can be derived exclusively by means of the pure general character of the concepts themselves (with recourse to certain mental representations or images in the memory, which provide the basic building materials, so to speak, out of which the entire image is constructed).

With reference to Goethe, Steiner draws attention to the point that only a very restricted role is assigned to the application of mathematics in the process of the acquisition of knowledge in the natural sciences while, on the contrary or rather supplementarily, in this process, mathematical *method* is of decisive importance. In order to see this more clearly, it is first necessary to describe more precisely the relationship of mathematics with the natural sciences. The fact that every law, in so far as it gives rise to phenomena in the material world, has a mathematical aspect makes mathematics the seemingly most universal means of acquiring knowledge in the natural sciences in general. However, this universality is, at the same time, one of the weaknesses of mathematics. Its laws are too comprehensive and its contents too meagre to allow specialisation to the particular principles of any perceptible object of knowledge. Furthermore, in no case does the mathematical aspect embrace the fact itself.

In connection with a discussion on Goethe's relation to mathematics Steiner describes the situation with the following words: "The field of study of mathematics is size, what can be more or less. Size is, however, something that has no independent existence. There is nothing within the realm of human experience that is *only* size. Besides other characteristics every thing also has those which are determined by numbers. Since mathematics is concerned with size, it has as its field of study no perceptible object complete in itself, but only everything that can be measured or counted in connection with size. It separates off everything that can be arrived at by measuring and counting. In this way, it obtains an entire world of abstractions within which it then operates. It does not deal with things but only with things to the extent that they have size. One must admit that mathematics examines only *one* aspect of reality and that reality has many other aspects for which mathematics has no importance. Mathematical judgements do not fully describe real objects and are only valid in the hypothetical world of abstractions. We ourselves have conceptually separated mathematical validity, as *one* aspect of reality, from the real world and confined them to the hypothetical world of abstractions. Mathematics makes the size and number of things abstract, establishes the entirely hypothetical connections between sizes and numbers, and thus floats in a pure world of thought. The things that exist in reality, in so far as they have size and number, permit the application of mathematical truths. Therefore, it is very definitely a mistake to believe that one can encompass all of nature with mathematical judgements. Nature simply is not only quantity (quantum); it is also quality (quale), and mathematics deals only with the former. The mathematical method of describing objects and the purely qualitative method of doing the same must necessarily work hand in hand together; these two methods will come into contact with things where each of these methods will examine *one aspect*." (Steiner, 1883-1897, Chapter XII, p. 239f.)

I should like to make two observations on this definition of the field of study of mathematics. If one interprets the concept of "size" in the sense given to it by Hegelian logic, and the formulation employed by Steiner reflects this line of interpretation, then "size" has a much wider meaning than one may be at first willing to accept. The deciding point here is primarily Hegel's concept of measure (qualitative quantum), within which quality and quantity are fused together into an *immediate* unity. In this way Hegel gives expression to the fact that a simple change of the quantum (determined or limited quantity) can be an indication of a change in quality. On the other hand, "enumeration" and "measurement" may be interpreted not only in the ordinary (elementary arithmetic and Euclidean) sense since these activities represent a lower-level part of mathematics - one leaves this elementary area as soon as one does algebraic number theory and projective geometry - and, in a similar way, nature's quantitative aspect, in the sense given to it by Hegel, is not restricted to what is simply countable and measurable. These brief considerations are all that can be noted here about this interesting topic, but it must be clearly emphasised that it would be extremely fruitful for an understanding of the essential nature of mathematics to pursue all these various connections in detail.

In Steiner's opinion, mathematics is valued too highly if one defines and restricts real knowledge only to what can be described mathematically. Such an overvaluation does not do justice to the essential nature of mathematics, it even has a contrary effect: "It is exactly those people who protest against this overvaluation of mathematics that can first truly appreciate pure crystal-clear research which in the *spirit* of mathematics proceeds even where mathematics comes to a halt. The reason for this is that mathematics in its immediate signification deals only with what is quantitative. The realm of mathematics ends where the qualitative starts. – In this case, however, it is imperative to do research also in the field of qualitaties in the spirit of the mathematical method. *Goethe* strongly objected to an overvaluation of mathematics. He did not want the qualitative to be bound by a purely mathematical mode of treatment. But he desired that all thinking be done in the spirit of mathematics, following the pattern and example of mathematics." (*Steiner*, 1904, p. 16f)

"But one can only express mathematically that which is totally comprehended by space and time, that which has extension in this sense. As soon as one ascends to higher worlds where *extension* in this sense is not the only thing that is being treated, mathematics is of no use in its immediate form, which is under discussion here. But one must not err concerning the *manner* of looking at ideas (intuition), which is a fundamental aspect of mathematics. We must develop the capacity to speak about what is living, what is spiritual and similar things in a way that is just as free from and just as independent of a single observable figure as we speak about a circle independently of the single circle drawn on a piece of paper." (Steiner, 1904, p. 11)

When Steiner says that mathematics in its *immediate* signification deals only with the quantitative, apart from the signification of the *mode* of looking at things which is basic to mathematics, this statement leaves open the possibility that mathematics has the means to describe certain aspects of something that is qualitative as long as one pays attention to the fact that one is dealing also in this case with only *one* side of the thing, namely, with its quality, in so far as this qualitative thing appears in space and time. An example of this is the mathematical characterisation, which will be mentioned in what follows, of the concept introduced by Steiner for the description of certain phenomena within the world of formative forces, namely, a "qualitatively opposite space" or "counterspace", as it was also called (*Steiner*, 1921a), with the help of projective geometry.

We shall now turn our attention to the other above-mentioned aspect of the training of modern consciousness by means of mathematics. Once one has recognised that consequent mathematical thought is *pure* thought, all considerations which have a connection with pure thought as a means for the development of and as a organ of a supersensual mode of experiencing things can be related to consequent mathematical thought (for this latter point, see *Steiner*, 1922, Lecture of 10th April).

Steiner refers very clearly to this connection: "On account of the fact that one with mathematics lives in the area of free creative spirit, with the help of mathematics the spirit's substantiality can be seen most clearly in inner self-knowledge. If one directs the mode of experiencing things from the figures which one constructs in the course of mathematical activity back to the activity itself, one will then become fully conscious of what it is that one is doing since one lives in a sort of free creative spirituality. – One must then only add the flexibility of the soul in order to extend the very same creative inner activity which one unfolds in mathematics to other areas of inner experience. In this flexibility of the soul lies the power to ascend to imaginative, inspired and intuitive knowledge." (*Steiner*, 1923, p. 151)

In a public lecture delivered to an audience of university graduates about the "Place of Anthroposophy in the Sciences", Steiner illustrates some results of his research in the spiritual science, which are relevant to mathematics and sums up his remarks with the following comments: "Correct understanding of the mathematical frame of mind leads us directly to the concept of clear-sighted or supersensual learning and experience. (...) One who wants to understand the clear-sighted process must search for it where it is available in its most primitive state: in the form of mathematics. (...) And if we succeed in training our mathematical mind by means of exercises, then we shall not only see spatial relationships in the world around us, but we shall learn to see spiritual beings as well as to recognise spiritual beings that manifest themselfs to us right up to their inner spiritual profoundness (...), if we in this way transpose to higher regions what we practise in mathematics." (Steiner, 1922, Lecture of 8th April, p. 41)

But Rudolf Steiner not only linked mathematics to spiritual science in the anthroposophical sense, he also gave concrete proposals for a spiritual complementation and permeation of the natural sciences with the help of selected mathematical concepts and methods. It is not possible here to discuss in detail these suggestions, which are scattered over a large number of lectures. However, one thing must stand out, namely, that Steiner repeatedly referred to the importance of synthetic projective geometry (see *Whicher*, 1970, Chapter IX). In the course of a public conference for university graduates he brought attention to the point that there is an analogous relationship between the method of *synthetic* projective geometry compared to the method of *analytical* geometry on the one side, and the method which should be used by the natural sciences for apprehending the laws of living beings compared to the method that simply describes dead forms in a formal mathematical way on the other side.

"(...O)ne must clearly realise that one can never arrive at essential inner distinctiveness by means of mathematics if one conceives of mathematics in the restricted sense in which it still today is often understood.

But we can already see in mathematics a sort of way that leads out of mathematics itself. From what I have already said, you can infer that this way, which leads out of mathematics, must be similar to the way which we traverse when we, with a kind of mathematics that is entirely figurative, uninvigorated and inefficacious, with this figurative kind of mathematics, dive down into the living and life-spending nature. There we are diving down into something that in a certain manner starts with our free mathematical activity and forces mathematical rules into an event that is force-full or living in itself, that is something in itself and about which we have to say: it is not completely accessible by means of mathematics; the thing maintains its essential independence and its essential interiority over and against the inner transparency of mathematics.

This way, which is travelled when one simply searches for the transition from the unreal mathematical mode of thought to the real scientific mode of thought, can in a certain manner already now be found within mathematics itself. And we see how it can be found if we consider internally and not externally the efforts which thinking has made in the transition from simply analytical geometry to projective or synthetic geometry, as the most recent advances in knowledge envision it." (*Steiner*, 1926, p. 68f.)

One should note that in Steiner's view the mode of thought employed by synthetic projective geometry "leads out of mathematics". This throws an entirely new light on his above-mentioned characterisation of mathematics and its lack of efficacy in respect to an experience of the effective qualities of organic nature. From the above quotation it is clear that synthetic projective geometry is considered by him to be a means of experiencing (at least a part of an area of) just this effectiveness or living-forces of nature. Synthetic projective geometry is provided as an example of where mathematics escapes from formal restrictions and develops into a differentiated organ of cognition, which signifies an initial step towards imaginative perception: "If one really follows with his inner soul the way which leads from analytical geometry to synthetic geometry and if one sees how one there is, I would like to say, caught up by something which already approaches reality as this reality is present in the actual outer world of nature, then one has the same inner experience, exactly the same inner experience, that one has when one ascends from the usual concepts of comprehension and from ordinary logic to the imaginative. One only has to go further on in imaginative perception. But one has got a start when he begins the transition from analytical to synthetic geometry. One finds that he is caught up there by what results from determination through outer reality, by which one has grasped the result, and one feels that in the same manner in imaginative perception." (*Steiner*, 1921, p. 80f.)

At this point it is possible to have the impression that synthetic projective geometry represents "only" a particularly suitable means of exercise for the development of the first supersensual level of cognition, of the imagination. This is not so. Synthetic projective geometry absolutely has the capability of penetrating into the essential effectiveness of natural events: "And this new synthetic geometry is just the way to get out of purely formal mathematics and come to the problem of where one has to apprehend the empirical (...) and to show how one has to conceive of mathematics in nature from the inside". (*Steiner*, 1922, Answers to Questions (Fragenbeantwortung), p. 154)

How and for what areas this happens in a real sense is another question. As a concrete suggestion for research in this direction, it will be useful to examine the following observation which Steiner made while answering questions during a course for university graduates in The Hague in 1922: "One is really dealing with here, as long as one (...) keeps in sight the roots of the plant, one is dealing with a special development of gravity. There one is strictly within the ordinary dimensions of space. But if one wants to explain the shape of the flower, then (...) instead of taking the initial point of the co-ordinates, one has to take infinitely distant space; which is really only another form for the point. And then one has a situation where, instead of going outwards with a centrifugal motion, one goes inwards with a centripetal motion. (...) Instead of the thing spraying out, it pushes inwards from the outside, and then one gets movements that are slipping or scraping movements or movements with pressure, for all of which it would be a mistake to take the axis of the co-ordinates from the central point of the co-ordinates, where one has to take the infinite sphere as the central point of the co-ordinates, and then only co-ordinates leading to the centre. Therefore, one gets a co-ordinate system that is qualitatively opposite as soon as one comes into the ethereal." (Steiner, 1922, Answers to Questions (Fragenbeantwortung), p. 153)

George Adams (1894-1963) and Louis Locher (1906-1962) succeeded independently of each other in providing mathematical interpretation of these and other introductions of Steiner to geometry of a "qualitatively opposite space" or "counterspace" (see *Steiner*, 1921a, Lecture of 15th January; *Steiner*, 1922, Lecture of 9th April) within the limits of one of the projective metric, that is, non-Euclidean geometries. The geometry in question in this case is one which is completely dual in respect to Euclidean geometry and which was mentioned above as "polar-Euclidean geometry" (It is also known as "dual-Euclidean geometry.). The first attempts at a concrete and detailed development of applications of these thoughts on the general principles of the growth and the form of plants can be found in *Adams/Whicher*, 1960 as well as in *Edwards*, 1986.

From the point of view of the potential existence of a space that is opposed to Euclidean space, the purely mathematical research carried out in the nineteenth century on the basic concepts of a non-Euclidean mechanics acquires a completely different significance. Could this research not permit the development of a concept of force which would provide elucidation of the working of "qualitatively opposite", "peripheral" or "ethereal" forces in the field of mechanics? This question led George Adams to introduce the concept of "peripheral universal forces" (in opposition to the punctual central forces of ordinary mathematical physics) and to examine it more closely in respect to its significance in mechanics (*Adams*, 1956-59). In doing this he could base himself on the earlier work of Klein, Lindemann and others because they had already brought together everything that was essential concerning pure mathematics (see *Ziegler*, 1985). This is not the place to examine this research in detail. However, it should be mentioned that this work has proven to be an important initial stage for far-reaching new interpretations of mathematical physics in the light of projective metric geometry (see *Gschwind*, 1977, 1979, 1985, 1986) and, furthermore, has demonstrated in more than exemplary fashion the fruitfulness of synthetic geometry and of the non-Euclidean geometry embedded in it for an interpretation of mathematical physics in accordance with anthroposophical spiritual science. Many questions connected with this new work are, of course, still awaiting definite formulation and answers as well as precise treatment of all the details involved. This research has only just begun, both in respect to the philosophical and spiritual aspects and in respect to the natural sciences and mathematics.

Just as the discovery of non-Euclidean geometry made an important contribution towards helping the formal mathematical method achieve a breakthrough and establish its pre-eminence in the natural sciences, it also seems to open up real possibilities for mathematical physics, which includes certain parts of fields regulated by invigorative principles of nature as well as a mathematics of living organisms. One of the main purposes of this article is to bring attention to this twofold aspect of the importance of non-Euclidean geometries for the development of scientific method.

10. Conclusion

The recognition at the beginning of the nineteenth century of the existence of a non-Euclidean geometry, thanks to Gauss, Lobachevskij and Bolyai, based on purely axiomatic (conceptual) considerations is considered as a symptom of faith in the truth content of pure thought, which was also expressed by German idealism though in a different way. A short time later this faith extended only to mathematical contents and consequently made the mathematisation of the natural sciences and the rejection of romantic natural philosophy necessary in accordance with currents of thought prevailing at that time. The integration of the non-Euclidean geometries into mathematics, which began in the sixties of the nineteenth century, played a part in the definitive separation of mathematical and especially geometrical abstraction from a way of looking at things determined by sensory perception. Once that mathematics had become conscious of its spiritual (abstract) strength, it began a decisive campaign of conquest throughout all branches of knowledge. On the other hand, because of this development, possibilities, which had not existed before to the same extent, were created for the training of pure thought and of supersensory imagination with the help of mathematics.

Rudolf Steiner was responsible for a new understanding of mathematics as preparatory training for spiritual comprehension as well as a methodological principle for the acquisition of knowledge in the natural sciences. Steiner threw clear light on the possibilities and limits of mathematics. Its strong point in developing consciousness lies in its methodological purity and universality, its weak point is its limitations in describing sensory phenomena because of the special form of their contents. Certain contents and methodological principles of mathematics, however, point towards areas beyond themselves, for example, synthetic projective geometry and the non-Euclidean geometries embedded in it, and, thus, these disciplines offer prospects of a mathematics of invigorative principles (mathematical physics and mathematics of living organisms).

Non-Euclidean as well as synthetic projective geometry plays an important role during a stage in the evolution of mathematics as it frees itself from the way of looking at things determined by sensory perception and becomes refined by complete abstraction, and also in the evolution of the natural sciences, which subsequently restrict themselves to the formal mathematical method. On the other hand, these geometries at the same time prove to be a powerful instrument for training pure thought and imagination free of sense perception. Thus, they are a means for leading man back to the invigorative reality of nature and spirit.

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Bibliography

Adams, G. (1933): Von dem ätherischen Raume. Stuttgart 1981 (2. Auflage).

– (1956-59): Universalkräfte in der Mechanik. Mathematisch-Astronomische Sektion, Dornach 1973.

Adams, G./ Whicher, O. (1960): Die Pflanze in Raum und Gegenraum. Elemente einer neuen Morphologie. Stuttgart 1979 (2. Auflage).

Bonola. R./ Liebmann, H. (1908): Die nichteuklidische Geometrie. Historisch-kritische Darstellung ihrer Entwicklung. Leipzig/Berlin.

Degen, H. (1956): Die Naturforscherversammlung zu Berlin im Jahre 1828 und ihre Bedeutung für die deutsche Geistesgeschichte. *Naturwissenschaftliche Rundschau*, Bd. 9, S. 333-340.

Edwards, L. (1986): Geometrie des Lebendigen. Vom Erleben gestaltbildender Naturkräfte. Stuttgart.

Goethe, J.W. (1793): Der Versuch als Mittler zwischen Objekt und Subjekt. Goethes Naturwissenschaftliche Schriften in Kürschners Deutscher Nationallitteratur, Bd. 2, (1887) = R. Steiner GA 1b, Dornach 1975.

Gschwind, P. (1977): Der lineare Komplex – eine überimaginäre Zahl. Dornach.

– (1979): Methodische Grundlagen zu einer projektiven Quantenphysik. Dornach.

– (1985): Zur Deutung mathematisch-physikalischer Strukturen. Math.-Phys. Korrespondenz Nr. 139, S. 18–25.

– (1986): Raum, Zeit, Geschwingidkeit. Dornach.

Hegel, G. W. F. (1830): Enzyklopädie der philosophischen Wissenschaften im Grundrisse (Dritte Ausgabe). Theorie-Werkausgabe Bd. 9, Suhrkamp, Frankfurt 1970.

– (1831): Wissenschaft der Logik (Zweite Ausgabe). Theorie-Werkausgabe, Bd. 5, Suhrkamp, Frankfurt 1969.

Henderson, L. D. (1983): The Fourth Dimension and Non-Euclidean Geometry in Modern Art. Princeton.

Jammer, M. (1980): Das Problem des Raumes. Darmstadt.

Kant, I. (1781): Kritik der reinen Vernunft.

– (1786): Metaphysische Anfangsgründe der Naturwissenschaft.

Klein, F. (1926): Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert. Teil I. Berlin.

– (1928): Vorlesungen über nicht-euklidische Geometrie. Berlin.

Locher, L. (1937): Urphänomene der Geometrie. Dornach 1980 (2. Auflage).

– (1940): Projektive Geometrie. Dornach 1980 (2. Auflage).

Mainzer, K. (1980): Geschichte der Geometrie. Mannheim/Wien/Zuerich.

Pont, J.-C. (1986): L'Aventure des parallèles. Histoire de la géometrie non euclidienne: Précurseurs et attardés. Bern/Frankfurt/New York.

Reichardt, H. (1976): Gauss und die nichteuklidische Geometrie. Leipzig.

Steiner, R. (1883-97): (Einleitung zu) Goethes naturwissenschaftlichen Schriften. GA 1, Dornach 1973.

– (1886): Grundlinien einer Erkenntnistheorie der Goetheschen Weltanschauung. GA 2, Dornach 1960.

– (1894): Die Philosophie der Freiheit. GA 4, Dornach 1978.

– (1904): Mathematik und Okkultismus. In: Philosophie und Anthroposophie. GA 55, Dornach 1965.

– (1921a): Das Verhältnis der verschiedenen naturwissenschaftlichen Gebiete zur Astronomie. 15. Vortrag vom 15. Januar. GA 323, Dornach 1983.

– (1921b): Mathematik und anorganische Naturwissenschaften. Vortrag vom 5. April. In: Die befruchtende Wirkung der Anthroposophie auf die Fachwissenschaften. GA 76, Dornach 1977.

– (1922): Die Bedeutung der Anthroposophie im Geistesleben der Gegenwart. Dornach 1957.

– (1923): Goethe und die Mathematik. In: Der Goetheanumgedanke inmitten der Kulturkrisis der Gegenwart. GA 36, Dornach 1961.

Unger, G. (1940): Zur Charakteristik der wissenschaftlichen Methode: Nichteuklidische Geometrie. Das Goetheanum, Bd. 19, S. 43-45.

Whicher, O. (1970): Projektive Geometrie. Stuttgart.

Ziegler, R. (1985): Die Geschichte der geometrischen Mechanik im 19. Jahrhundert – eine historisch-systematische Untersuchung von Moebius und Pluecker bis zu Klein und Lindemann. Stuttgart (Boethius, Bd. XIII).

Additional Bibliography (1997)

Adams, G. (1996): Universalkräfte in der Mechanik. Perspektiven einer anthroposophisch erweiterten mathematischen Physik. Dornach (Mathematisch-Astronomische Blätter – Neud Folge, Bd. 20).

Bonola, R. (1906): La geometria non-euclidea: esposizione storico-critico del suo sviluppo. Bologna.

Edwards, L. (1993): The Vortex of Live. Nature’s Patterns in Space and Time. Edinburgh.

Gschwind, P. (1991): Der lineare Komplex – eine überminaginäre Zahl. Zweite umgearbeitete und erweiterte Auflage. Dornach 1991 (Mathematisch-Astronomische Blätter – Neud Folge, Bd. 4).

Rosenfeld, B. A. (1988): A History of Non-Euclidean Geometry. Evolution of the Concept of a Geometric Space. New York (Studies in the History of Mathematics and Physical Sciences, Vol 12).

Steiner, R. (1904–1922): Die vierte Dimension. Mathematik und Wirklichkeit. Hörernotizen von Vorträgen über den mehrdimensionalen Raum und von Fragenbeantwortungen zu mathematischen Themen. GA 324a, Dornach 1995.

– (1922): Damit der Mensch ganz Mensch werde. Die Bedeutung der Anthroposophie im Geistesleben der Gegenwart (Haager Hochschulkurs). GA 82, Dornach 1994.

Ziegler, R. (1992): Mathematik und Geisteswissenschaft. Mathematische Einführung in die Phiosophie als Geisteswissenschaft in Anknüpfung an Plato, Cusanus, Goethe, Hegel und Steiner. Dornach.

– (1993): Goethes Ideen zur Mathematik. Materialien zu Goethes Mathematikverständnis. Dornach.

– (1995): Selbstreflexion. Studien zu Selbstbeziehbarkeit im Denken und Erkennen. Dornach.

Dr. Renatus Ziegler
Mathematisch-Astronomische Sektion
Goetheanum
CH-4143 Dornach