Mathematics as a spiritual science
Philosophical investigations into the significance of mathematics with reference to Plato, Goethe and Steiner

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Introduction and overview

Mathematics is one of the necessities of modern life. There is hardly an aspect of life in which mathematics does not play a more or less significant part via scientific and technical achievements. Mathematics occupies an increasingly important position even in the everyday life of university humanities departments.[1] A characteristic of this is that people *apply* mathematical concepts to areas outside mathematics without in each case taking into detailed account its deeper significance.

Precisely because mathematics has become an almost all-pervading instrument of scientific and technical process, there is at the present time a need for a consideration of the subject's inner nature, its possibilities and its limitations. For instance, the important question arises in ordinary cultural life of whether there may be some ways of dealing with mathematics, hitherto seldom considered, which could be cultivated alongside both the development of pure mathematics as well as its extension to the mathematical models of the applied branches. Mathematics by and large serves and has served the private or institutional acquisition of knowledge, or it is studied as an indispensable body of knowledge for getting to grips with the demands of life in modern occupations; ultimately it is used above all as an instrument for the progress of modern civilisation. If mathematics should not just be useful, but of real significance for deepening human *culture* and *education*, then other ways of cultivating mathematics must be sought.

In presenting this unusual approach to mathematics I shall tie in with Plato and Goethe[2]. However, these authors serve only as a point of departure for an investigation independent of this connection.

Mathematics was for Plato a means of diverting the soul from contemplating the objects of the senses to becoming aware of the spiritual ground of existence. Mathematics itself cannot give information about the divine, but it can prepare the soul for beholding it (theoria). Can a start be made with this view nowadays? Are Plato's comments to be understood merely in the sense of a myth or do they still, or once again, have a real foothold in the potential experience of people today?

With reference to Plato, Goethe saw in mathematics first and foremost a training instrument which leads people to exactness and methodical certainty in the process of cognition. No other science leads to such complete certainty of method and content as mathematics. Thus Goethe was aware that in cognizing the world it is not only a matter of *applying* mathematical content, but also of practising the *mathematical method.* In addition, the fundamental occupation with mathematics leads according to Goethe (1792) to experiences 'of a higher kind', which are connected with the development of an organ the objects of which do not belong to the material world.

I will show here that both Plato's and Goethe's points of view can be concretely related to the views of modern mathematics. For this we shall turn first of all to the concept of symmetry which plays a prominent part in pure mathematics, in classical and modern physics, in other sciences and in philosophy.

By mathematical examples it will be shown that by looking for symmetry it is a matter of *invariant structures*, that means properties which are not subject to change. At the basis of every change, whether or not it takes place in time, there is a *principle* of transformation, which itself does not change. This principle is an invariant structure in the flow of change. It is that concrete principle according to which the way the change takes place is determined. From the standpoint of the cognizing subject, such a principle is needed as a conceptual standpoint in order to be able to grasp changes at all.

Principles or structures found in this way belong to a realm which, as I shall show, lies beyond all changes. It can be referred to as the realm of *ideas* or *laws*.[3] In this sense, mathematics belongs to the spiritual sciences or humanities, because they are concerned with a content which only manifests by means of the thought activity of the human spirit. This realm is related to the realm of forms in the Platonic sense. The Platonic forms however have an additional property. They are ideas at work in nature.

This difference between an idea present in individual thought (concept) and an actively creating form at work in nature surfaced in the Middle Ages in the term *universale post rem* and *universale in re*. These are to be distinguished from the self-existent and self-supporting *universale ante rem* which unfolds its effectiveness out of nothing but itself. Nominalists deny at least the existence of universals at work in the phenomena, and often even the universally objective nature of the idea. Realists of ideas on the other hand are of the opinion that ideas not only have an *objective existence,* but also an *immanent effectiveness*.[4]

In dealing with mathematical laws, their universal objective nature usually poses one no problem. There is however also the possibility of properly demonstrating the *effective* nature of the idea, that is to say the concrete constitution which is active in the form or being. This is done by starting from the mathematical thoughts which are actually at work. Thus mathematics can be a point of departure for a science of the spirit *as it is active*, thereby supplementing traditional spiritual science, which is a science of *products* of the spirit which have arisen in the past and stay in existence after the spirit is active. This science of the currently-active spirit must meet Goethe's requirement as regards *method*, thereby extending the realm of objects, however, to a content Plato referred to which is no longer accessible to the usual senses. This is the particular task of the anthroposophical spiritual science developed by Rudolf Steiner.

1. Plato: Mathematics between form and image

In nature Plato distinguished the objects and processes from the living principles which produce them. The former are the *images* which come into existence and remain susceptible to change and the latter are the creative principles, the *forms*, eternally existent and ever the same.[5] This differentiation according to *objects of cognition* corresponds to a distinction in *methods of cognition*. Images appear to human consciousness in the form of ready made concrete mental representations or judgements (doxa), whereas the forms involve the living cognitive process of reason, the intuition of ideas (theoria). Between these two kinds of experience lies the intellectual cognitive process concerned with the objects of science, its abstract concepts and ideas. Mathematical thinking belongs especially to this domain. Mathematics shares with the rest of science the property of being ultimately derived from preconditions (postulates, axioms) which cannot be deduced (proved) out of themselves.

However, the objects of mathematics are not images, because mathematical concepts are not concerned with the specific properties of *single* objects, but with structures to which a whole class of objects belongs. For instance, in determining the concept circle it is not a matter of including in this concept the position of the centre or the length of the radius of any particular circle, but of singling out a general principle which forms the basis for *all* circles. Although such a principle can then fit all circles, it is not the only one which has this property. Thus for instance, the following definitions of a circle are equivalent to each other to the extent that each circle in the sense of one definition is a circle in the sense of the other and vice versa:

Distance definition of a circle

A circle is the geometric set of all points P of a plane which have the same distance from a given point M of the plane (Fig. 1)

Right angle definition of a circle

A circle is the geometric set of all points P at the foot of the perpendiculars from point S of a plane to those lines of the plane which pass through point R $\neq$ S (Fig. 2

Proof of the equivalence is shown immediately by Figure 3 which shows a special case of the given properties, a rectangle inscribed in a circle. If K is a circle in the sense of the distance definition and if P,Q and R,S are pairs of points which lie on K, then $|RM| = |MS| = |MP| = |MQ|$. The quadrilateral RPSQ is rectangular because its component triangles are isosceles triangles. Thus K is a circle in the sense of the right angle definition. If on the other hand K is defined in the latter sense, then one can select point M on RS so that $|RM| = |MS| = r$; point Q is the foot of the perpendicular parallel to PS through R. Thus the

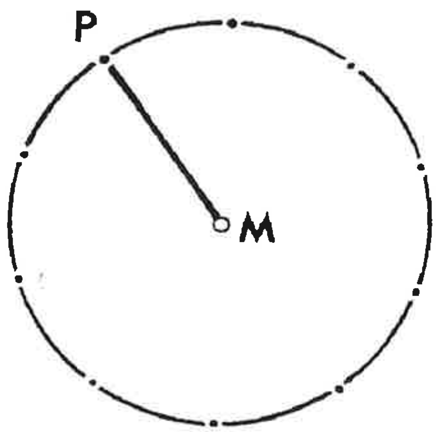

Figure 1

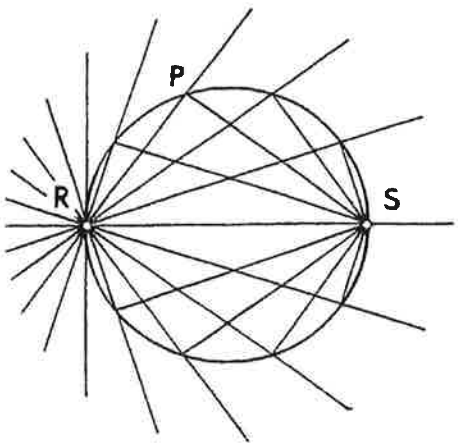

Figure 2

quadrilateral RPSQ is rectangular and it follows that $|PM| = r$ for all points P at the foot of perpendiculars.

A mathematician can find many other equivalent definitions of a circle, thus revealing many insights into how the circle principle fits into the framework of geometric concepts. The higher unity of all these circle principles (definitions), the general structural principle or *law of the circle* that underlies them is not itself an immediate object of mathematics. It is presupposed by mathematics and occurs in it

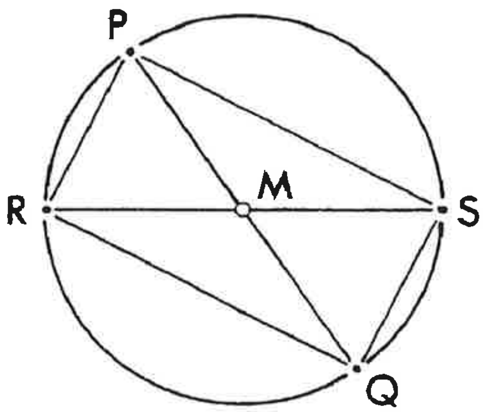

Figure 3

only through the mediation of particular conceptual conditions. The law of the circle always appears in mathematics in already concrete relationships, for instance to certain geometric concepts such as distance, right angle etc. on the basis of axioms which likewise are presupposed.

In this sense, the objects of mathematics are not self-supporting and self-sufficient forms. On the one hand they are based on presupposed axioms and on the other they reflect the particular conceptual context of the components of the relevant definitions.

From the self-supporting ideal *content* of the Platonic form, which is the superior structural principle of all images, it is necessary to distinguish that dynamic and creatively real *effectiveness* which is active in producing concrete images and really, i.e. not only ideally, underlies their existence. The philosopher's schooling in the sense of the *Republic* (Book 7) has as its aim his preparation for grasping in cognition the creative forms. By means of a schooling in the 'mathematical sciences' (arithmetic, geometry, harmonic theory and astronomy) the philosopher's soul was attuned to beholding the forms. However, as Plato wrote in his *Seventh Epistle* (342a-344b), these sciences are not immediately appropriate for grasping the highest objects of knowledge, the creative forms. But practising them prepares the ground or develops the faculty for being able to set eyes on their creative quality.

Plato did not detail this path (problem of the 'unwritten theory') because he trusted that those who see through the problem of cognition of the forms would also find their way to them.

2. *Goethe: Extent and limits of mathematics*

What Plato indicates, Goethe clearly expresses: Grasping the laws of nature requires a corresponding organ, a kind of 'higher experience within experience'.[6] With this, Goethe extends the domain of phenomenology to an area hitherto excluded from it. Not only is an experience or phenomenon grasped only with the senses valid, but now also one produced through thinking. For Goethe, the latter is not beyond the bounds of nature but within it. Insight into the lawful working of nature can be achieved by careful contemplation of the phenomena. This means through the development of ideas, which reveal that self-supporting essence which remains unchanged in relation to the whole diversity of the natural phenomena and of the experiments people perform. These ideas are the universal principles which structure every individual phenomenon.

The ability to develop and perceive ideas can be cultivated especially through mathematics. For the aim of mathematics is directed to showing not individual examples but general principles. It can therefore serve to train *pure intuition* divested of all specific sensory elements. It is this faculty which is necessary for grasping natural laws. On the other hand, as Goethe pointed out in his essay 'The objective and subjective reconciled by means of the experiment' (1792), practising mathematics also helps one to learn a methodical discipline which exhibits a firm basis for knowledge of nature. With respect to both these areas, the mathematical approach in Goethe's view is of utmost importance. His oft cited backwardness in relation to mathematics concerns the application of the content of mathematics and not on the subject per se or the mathematical method. Goethe was however not in principle against the application of mathematical content to the processes of nature. He even wished it for extending his own researches. However, he discovered many misuses in this field, especially a concomitant restriction of outlook to quantitative relations to the exclusion of qualitative aspects.

As Goethe understood it the first and foremost task of mathematics is to serve as an instrument for the clear structuring of scientific thinking in order to work out in a surveyable and clearly organised form the invariant structures or ideas which correspond to the phenomenal world. According to Goethe these structures themselves have an experienceable character. Following on from Goethe (in the absence, to my knowledge, of Goethe having expressed this explicitly in such a form), it may be asked whether ideas are merely invariant relative to ordinary experience and nevertheless share with ordinary experience the property of changeability or whether in their essence they are also invariant relative to individual consciousness. The problem arises of whether the 'higher experience within experience' can be investigated also in the same exact and experiential manner as both the mathematical and the phenomenological methods demand in their application to the objects of nature and whether their constituent invariant properties can be discovered.

3. Symmetry and invariance

In this section we shall look at mathematics itself and the activity it involves. We shall not go into any recorded observations *about* mathematics, but instead develop the relevant insights from handling mathematics.

In the examination of the concept of symmetry in the sciences it emerges that it is difficult to unite the various meanings of 'symmetry' under a single viewpoint. But two aspects can be distinguished which are to be found in almost all approaches to the clarification of this concept. On the one hand there is the conception of symmetry suggested by mathematics as an *invariance* with respect to certain transformations or changes, and on the other hand the practical significance of *symmetry-breaking* or asymmetry.[7] The latter reveals itself on closer inspection as an expression of a higher symmetry or harmony; with this the subordinate symmetries are generally 'broken' or 'destroyed' by a transformation which leaves the higher symmetry invariant.

We shall consider an elementary example. The structural principle of a triangle contains three different points not lying on the same straight line as well as their connecting lines. As subordinate structures we can further distinguish acute angles, right-angled and obtuse-angled triangles according to whether all angles are less than, equal to or greater than 90° respectively. Each individual triangle has properties included by these structures. But it also has properties which do not immediately belong to the specified structure such as position as well as precise lengths of sides and angles. This is characteristic for the relationship of an object (thing) to its structure. The former has accidental properties additional to and not contained in the structure which embraces the essential properties, but which precisely distinguish it as a particular object.

The symmetry transformations of objects with a particular structure embracing the essential properties comprise those alterations of the objects which concern only the

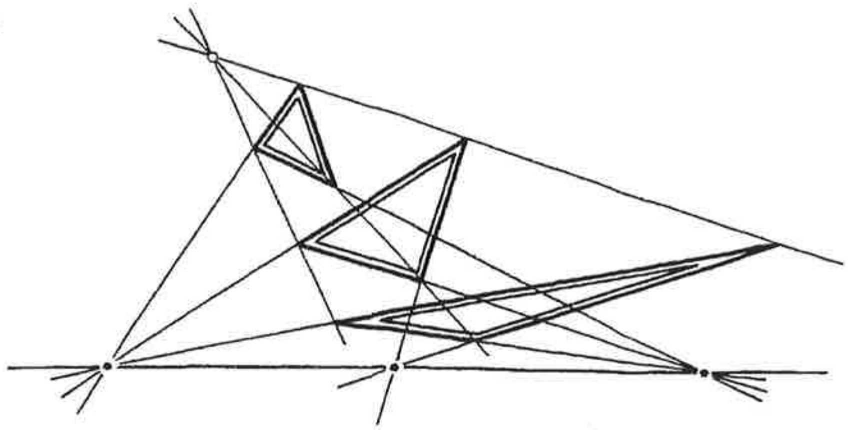

Figure 4

accidental properties. If in our example we stipulate, as the basis of the primary structural features, the above classification of the triangle, then the only symmetry transformations we are concerned with are the Euclidean congruence transformations and similarities. A transformation which converts a right-angled to an obtuse- or acute-angled triangle is, in relation to *these* structural features, a breaking of symmetry, in particular, a homology (Fig.4 shows such a transformation, also called perspective collineation). This is so, because the transformation does change the essential properties of this structure. But from the point of view of the mere structure of a triangle, those kinds of transformations are also symmetries because they leave the triangle as such invariant.

We can see that the changes or transformations mentioned in no way affect the structure as such, only the objects or things which are subsumed under this structure, i.e. which are phenomena of it. Also with symmetry-breaking the subordinate *structure* is not broken, but the *objects* which are part of the structure are changed.

Transformations of a set X, meaning a one-to-one correspondence between elements of X generally form a group. A *Group* **G** is a set G of elements with an operation defined on G ('multiplication'), which sends any two elements g_1, g_2 of G to an element g_1, g_2 of G, such that the following properties hold:

1) Associativity: $g_2(g_2g_3) = (g_1, g_2)g_3$;
2) Existence of identity: There is just one element e in G, such that $eg = ge = g$ for all g in G;
3) Existence of inverses: For each element g in G there is just one element g^{-1} in G, such that $gg^{-1} = g^{-1}g = e$.

An example of an finite group, meaning a group with a finite number of elements, is given by the group **S** of symmetries of an equilateral triangle *ABC*. These symmetries transform any such triangle only in its position not its structure.
Let s, s', s'' be the reflections on the three axes of symmetry of an equilateral triangle and r the 120° rotation anticlockwise around the middle point M (Fig. 5). The

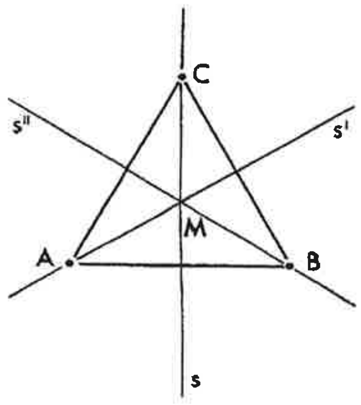

Figure 5

combination rs signifies that first r and then s is applied. Clearly then $s' = rs$ and $s'' = r^2s$. If we now introduce the identity operation e which leaves everything unchanged we have as elements of the group: $S = \{e, r, r^2, s, rs, r^2s\}$. Furthermore we have $sr = r^2s$, $r^3 = e$ and $s^2 = e$. With these formulae the table of all interrelations of the elements can be worked out (Cayley table):

	e	r	r^2	s	rs	r^2s
e	e	r	r^2	s	rs	r^2s
r	r	r^2	e	rs	r^2s	s
r^2	r^2	e	r	r^2s	s	rs
s	s	r^2s	rs	e	r^2	r
rs	rs	s	r^2s	r	e	r^2
r^2s	r^2s	rs	s	r^2	r	e

With a little consideration it should be clear that this group is structurally similar (isomorphic) to the group **P** of transformations, or *permutations* (rearrangements) of an finite set of three elements. If for example we take the first three natural numbers 1, 2 and 3, then these can be arranged in six different ways:

$$\{1, 2, 3\}, \{2, 3, 1\}, \{3, 1, 2\},$$
$$\{2, 1, 3\}, \{1, 3, 2\}, \{3, 2, 1\}.$$

We are interested in those operations (transformations, rearrangements or permutations) by which from $\{1, 2, 3\}$ all other arrangements can be deduced.

Through the operation ρ of cyclic transposition, meaning the operation $1 \to 2$, $2 \to 3$, $3 \to 1$, the arrangement $\{1, 2, 3\}$ becomes $(2, 3, 1\}$. The transition from $\{1, 2, 3\}$ to $\{3, 1, 2\}$ is produced by applying ρ twice, i.e. by $\rho\rho = \rho^2$. The arrangements $\{2, 1, 3\}$, $\{1, 3, 2\}$ and $\{3, 2, 1\}$ are derived from $\{1, 2, 3\}$ by keeping one element fixed and switching the two others, i.e. through the operations:

$$\sigma: 1 \to 2,\ 2 \to 1,\ 3 \to 3;$$
$$\sigma': 2 \to 3,\ 3 \to 2,\ 1 \to 1;$$
$$\sigma'': 1 \to 3,\ 3 \to 1,\ 2 \to 2;$$

We understand by the operation $\rho\sigma$ the consecutive execution of the operations ρ and σ, thus the following can easily be verified: $\sigma' = \rho\sigma$ and $\sigma'' = \rho^2\sigma$. Further, $\sigma\rho = \rho^2\sigma$, $\rho^3 = \varepsilon$ and $\sigma^2 = \varepsilon$ where ε is the identity operation $1 \to 1$, $2 \to 2$, $3 \to 3$. Thus we obtain as elements of the group of operations (permutations) $P = \{\varepsilon, \rho, \rho^2, \sigma, \rho\sigma, \rho^2\sigma\}$. With these formulae the table of all relationships of group elements to one another (permutations) can be drawn up:

	ε	ρ	ρ^2	σ	$\rho\sigma$	$\rho^2\sigma$
ε	ε	ρ	ρ^2	σ	$\rho\sigma$	$\rho^2\sigma$
ρ	ρ	ρ^2	ε	$\rho\sigma$	$\rho^2\sigma$	σ
ρ^2	ρ^2	ε	ρ	$\rho^2\sigma$	σ	$\rho\sigma$
σ	σ	$\rho^2\sigma$	$\rho\sigma$	ε	ρ^2	ρ
$\rho\sigma$	$\rho\sigma$	σ	$\rho^2\sigma$	ρ	ε	ρ^2
$\rho^2\sigma$	$\rho^2\sigma$	$\rho\sigma$	σ	ρ^2	ρ	ε

By comparison with the symmetry transformations *r, s* and *e* of an equilateral triangle and the corresponding multiplication table, the group table above shows that these operations in fact have the same multiplicative structure as the permutations ρ, σ and ε of a set of three elements $\{1, 2, 3\}$.

From this it follows that with these two concrete groups we are dealing with realisations of one and the same structural principle, a so-called *abstract group*. In this, only the special multiplicative structure and not the concrete nature of the elements is considered. In addition, this is to be distinguished from the *notion of a group as such* given above which underlies all special abstract groups as a common structural principle.

The domain within which a variation takes place is thus in each case the domain of objects of the transformation, here meaning the set X in which the transformation operates. In the above example it was an equilateral triangle, a subset of elements of the plane, as well as a set of three elements {1, 2, 3}. Thus for every variation of a (not necessarily finite) domain of elements that conforms to a particular principle, i.e. a particular transformation, on the one hand there is something structural forming a basis which remains invariant through the transformation and on the other hand, all transformations of this kind generally form a concrete group which itself exhibits a higher structural principle for all transformations. This structural principle is in turn a particular case of an abstract group and the latter an example of a group.

4. Universal content and the individual's ability to experience mathematical laws

How can something experienced in individual consciousness have a universal character independent of this consciousness? This is the fundamental problem that is to be solved for the proof of the objective existence of mathematical laws.[8] To solve this, both direct and indirect methods have been suggested.[9] By *indirect methods* it is a matter of proving that without the acceptance of the reality of mathematical laws a meaningful and elegant science which is as plausible as possible to the human intellect would not be possible. Such *indispensability arguments* ultimately lead to *hypothetical* realism, a sort of myth about the reality of specified entities which *in this sense* cannot be distinguished from other myths, legends or creeds.[10]

By *direct methods* for the proof of the reality of mathematical laws, it is a matter of analyzing the immediate manner of experiencing these laws. Experience is part of individual consciousness. It is thus only accessible to introspection and for this reason has so far been rejected by many authors as suspect, unclear or unscientific. From the apparent failure of all attempts by means of introspection to come to objective results, in contrast to subjective enlightenment[11], it is almost exclusively the indirect method that is still taken seriously. In this essay it will be shown that the possibilities of the direct method are in no way exhausted or sufficiently researched - not to mention the fact that a consistent scientific consciousness can never and must never be satisfied with mere, albeit rationally-based, belief in a myth.

Before positive proof of the reality of mathematical concepts can be tackled, a few prejudices must be cleared out of the way.

First prejudice: The content and the process of mathematical thinking arise from convention. - The origins of conventions are not necessarily of a conventional nature: a 'convention' established for the first time cannot arise from an agreement, because it is initially known to nobody but the subject who establishes it. If it is possible however for this subject who establishes the convention to have an unconventional approach to thought, then is not clear why this should not be possible for other subjects too. In addition, agreements between people, which are communicated explicitly, inexplicitly or otherwise, require individual insight into or assent to the meaning of the agreement. Otherwise, in passing on conventions, one is merely dealing with blind faith or obedience.

Second prejudice: The subjective experience of mathematical thinking (introspection, intuition, inspiration etc.) is of an inexpressible nature and thus lies outside science. - Here is a confusion of thinking with communication, or rather the muddling of the content of thinking and the expression of this content in a language. In order to think, one neither has to talk to oneself nor communicate with oneself in any other way. In addition, the meaning of linguistic expressions cannot ultimately be inferred from a language; the investigation of the meaning always stops with the *individual* insight into the meaning of the expressions of the (natural) language.[12] Therefore, if what cannot be expressed in language cannot be exactly understood, then ultimately the source of knowledge of scientific investigation would be removed and thus science would only be able to be established through extra-scientific personal experiences.

Third prejudice: The experience of mathematical thinking belongs exclusively to the subject. It has no significance beyond the subject. - The determination of the subjective character of the experience of mathematical thinking occurs through the subject himself and results from the experience of his own activity which is connected with this experience as well as from the fact that only I myself experience directly what I think and no other person has an immediate part in my unspoken thinking. But this only means that the activity as well as the consciousness of the thought content belong to the subject; however this yields nothing about the constitution of the content. Here there often exists a further prejudice:

Fourth prejudice The subject produces the content of mathematical thought. - Not a single *direct* observation based on mathematical thinking has so far been advanced for this hypothesis. All phenomena which apparently support it concern the consciousness of contents, but not the contents themselves.

Fifth prejudice: The contents of mathematical thinking are determined through the structure of the psycho-physiological cognitive apparatus. - For immediate confirmation of this thesis it must be shown that for establishing and deducing mathematical laws the structural principles of the cognitive apparatus must of necessity be explicitly enlisted. In the direct experience of mathematical *thinking* (*not*: the formal-symbolical representation of this thinking) there are however no grounds for such an incompleteness or dependence of mathematics *in principle*. In addition, all arguments for the dependence of mathematical thought contents on the structure of the cognitive apparatus concern the *consciousness of* the contents, not these *contents* themselves. Finally there is the evident incompatibility and diversity of the contents of consciousness of mathematical thinking and the results of observation obtained by means of investigation of the cognitive apparatus.[13]

For a deeper insight into the structures of argument used here we shall introduce the distinction between proper and improper hypotheses. A hypothesis (model, theory, structure) with respect to a realm of facts is *improper*, when there are observations lying immediately inside this realm which justify the hypothesis. There must not merely exist inferences for confirmation of the hypothesis. A classical example for an improper hypothesis is the following statement: The period of swing of a freely swinging pendulum is dependent on the length of the pendulum.

A hypothesis with respect to a realm of facts is *proper*, when there are no immediate observations within this realm which justify the hypothesis. There exist only methods of inference which, from the factual material available, suggest the existence of something which is not itself part of this material. Any *indirect* method for the confirmation of realism is an example of this.

In the following investigation, strict attention will be paid to whether we are dealing with proper or improper hypotheses. This is of fundamental significance, because we are not dealing with the investigation of any arbitrary object, but with something which plays a fundamental role in all scientific activity, namely thinking, in particular in its strict form of mathematical thinking.

The natural sciences, especially physics, deeply depend on mathematical laws. If these are not to attract the criticisms of arbitrariness and inconsistency, the manner of experiencing mathematical thinking must itself be established inside the domain of science. In the sense of *naturalized epistemology*[14] this means that this experience should be traced back to processes, especially physiological ones, which can be understood scientifically. On more thorough inspection this shows itself to be a *proper* hypothesis, because nothing experienceable *within* mathematical thinking itself confirms it. In this connection, one should also take into consideration the discussion of the fifth prejudice.[15]

Therefore, it seems reasonable to investigate more closely the independent mode of experience appropriate to mathematical thinking (and to thinking in general). Following Gödel (1947/64) we shall call the process of insight in mathematical thinking 'mathematical *intuition*'. In using this term we are not committed in every respect to the details of Gödel's definition, but we shall by means of experiences of mathematical thinking itself develop more precisely in the following section what we think should be understood by it.

Gödel understands by mathematical intuition, not primarily an immediate knowledge, but a kind of forming of ideas by means of something immediately given. Gödel has not given a more exact definition of the function of or the elements of this intuition; his definition of the concept of intuition was thus challenged from various quarters and rejected as unnecessary.[16]

5. Mathematical intuition

Mathematical intuition must first be distinguished from *idea,* taken here in its usual sense as something that occurs to the individual mind (having an idea etc.). In previous sections we used this word with different connotations following Platonic tradition. Idea in its most common sense means, however, a content which is given to the thinking subject without him having contributed himself *directly* with his own conscious activity to the process of this content being given. Such ideas certainly play an important part in the life of a mathematician, but they come 'by chance' and are not subject to the control of individual consciousness. As a rule however, a fruitful idea is preceded by an intensive occupation with mathematical contents in the 'neighbourhood' of the contents of the idea. Furthermore, following the idea there is the task of finding the actual evidence, i.e. the concrete pattern and detailed interrelating of the contents of the idea with contents already known, for instance, axioms and theorems derivable from them.

What should be understood here by *mathematical intuition* are only those phases of the mathematical work by which the mathematician has a complete clarity and overview of his actions, i.e. where he knows exactly his point of departure and how he reached the contents he is actually thinking about. This implies no devaluation of other phases of mathematical thinking (heuristics, ideas, analogies, games etc.), but these are of a preparatory nature and are not determinative for the ultimately intuitive insight.

Mathematical intuition is bound by two conditions: One concerns the purity of the content produced in thought and the other the manner of its production. By purity of content we understand the complete freeing of mathematical thinking from concrete examples from the world perceived by the senses. Thus, in section 1 it was not a case of any particular circle existing anywhere, but of the principles which govern and constitute all circles.

The manner of production is concerned with the degree of comprehensibility and clarity of the insight into the inner necessity of a thought content being dependent on the extent to which the subject participates in the thinking process. We can comprehend completely only that which we ourselves bring about, bring into existence. Everything given without the subject's own activity is initially a problem for the attentive subject. In mathematical intuition, no content is given to the thinking subject without his having produced it. However, this does not mean that mathematical thinking itself produces its content (see previous section). Rather it means not only following every step of the process, but also *performing* these steps autonomously.

Inside mathematical intuition, two realms of experience can be distinguished from one another: one concerns the subject's activity (see following section) and the other the constitution of the content.

Within the process of mathematical intuition, three properties can be distinguished as regards the *contents of mathematical thinking*, i.e. the contents of mathematical concepts, here also called laws. These properties play a fundamental role in the judgement of the constitution, i.e. of the ontological make-up, of these contents. Attention has been drawn above to one of these properties, namely inner *necessity* and complete *comprehensibility* (see sections 1 & 3). Another concerns the *unchangeability* or *invariability* of the laws by the thinking subject. The laws offer a (passive) resistance to a corresponding test and cannot in their content be either changed or arbitrarily linked with other laws. Put metaphorically, mathematical thinking is 'guided' by the laws in maintaining its state of intuition - like someone's hand consciously feeling a marble relief. The relief does not press the hand, but it does not allow itself to be changed by it. Every apparently successful alteration of a law leads either to a new one or is confined merely to the concrete relationship of the subject to the thought contents. So-called extensions of concepts or conceptual generalisations (e.g. of the laws of multiplication) are not variations of a concept as such, but an expression of a different perspective of the thinking subject to the corresponding realm of laws.

The independent and self-supporting character of mathematical laws is revealed in mathematical intuition. They are, in fact, in the sense of section 3, invariants of the operations of individual mathematical thinking.[17] A structural principle higher than the operations carried out by the individual subject forms their basis. This is the universal principle of mathematical intuition used, indeed, by all mathematicians, but which none of them own privately.

Here the question arises as to whether mathematical laws are invariant only relative to the thinking subject, or whether they are generally (absolutely) invariant. The *invariance*[18] of laws means that their content cannot be subjected to a change by another being or by themselves. The invariance of laws implies their unchangeability or invariability, but the reverse does not hold true.

It must first be established that there is no experienceable, i.e. not only proper hypothetical, basis for the assumption of a variance or a changeableness of mathematical laws. What changes is at most the individual grasp of or the consciousness of these laws, but not the laws themselves.

The understandable psychological resistance to the invariance of laws is not primarily directed at mathematical laws, but at the acceptance of unchanging laws in general. This appears to be confirmed by so-called everyday experience. But here we do not make it sufficiently clear to ourselves that the acceptance in principle of a variance or changeableness of all laws has the consequence that there must be one or more 'super laws' which do not change and which exhibit with each concretely demonstrable change the structures which remain invariant (the invariants). For, given that law A transforms to law B, i.e. that A is changed in that it becomes B, the question arises: On the basis of which property can B be determined as coming from A? This is only possible when there is a predicate C which is common to both A and B, whereby B, as something still connected with A in some way, can be recognised as related with A. For this however, C must show an invariant property relative to the transformation of A to B, i.e. cannot be subject to change. Therefore the principle C is unchanging and A and B thus do not belong to the realm of laws.

It could be objected that here we are dealing with a proof of only *relative* variance or unchangeableness, but not one of absolute unchangeableness. That is not however the case, because the assertion behind this objection that all is relative is, taken in the absolute sense, necessarily self-contradictory.

From this it follows that the realm of change is not to be established in the realm of laws, but in the realm of phenomena, i.e. the place where these laws operate or take effect. The situation here is totally analogous to the relationship of abstract groups to the elements of their domain of possible transformations. The operations of the groups concern only these elements or sets of such elements (see section 3). - Sometimes the objection arises here that there might also be 'flexible' or 'living' concepts. Following the above discussion, this cannot mean a self-changeableness of concepts, but a flexible or living perspective of the thinking subject relative to the self-determined and unchanging contents of laws.

To conclude this section, attention should be drawn to the fact that it is not in the nature of the principle of mathematical intuition that it can only be used on mathematical contents. It cannot be denied from the outset that there are also concepts lying outside the domain of mathematics which can be manifested in the *form* of mathematical intuition.

6. Laws as active principles

In the previous section, two realms of experience in the process of mathematical intuition were indicated. One comprises the activity of the subject and the other the constitution of contents. We now turn to the activity. Firstly, in the process of mathematical intuition the focus of attention is directed to the content of thoughts. But in doing this a clear consciousness of the activity can also take place. This enables the transition from *naive* to *critical* thinking. In critical thinking one is conscious of the laws of ones activity, whereas in naive thinking, although one is active in accordance with these laws, the attention is exclusively devoted to the contents of thought. By critical thinking we do not therefore simply mean that the law of thinking is made the *content* of thinking. This is certainly necessary as a preparation, but for actual critical thinking it is insufficient.

If one has become aware of this law from observations of thinking and has clearly grasped it in thinking, then in subsequent acts of cognition one can consciously base the thinking process on it. This having actual hold of the law of thinking with regard to the thought content is critical thinking - and from now on only this critical mathematical thinking will be understood by the term *'mathematical intuition'*.

What comprises this law of thinking? It contains the requirement that only those conceptual contents will be considered as thought content which have been brought to manifestation by the conscious activity of the thinking subject. This concerns the components linked together in a concept as much as the connections themselves. The pure laws arising in the form of mathematical intuition in no way result from their own activity. They are totally passive yet nevertheless have an individual existence expressed by their invariability and invariance (see section 5). The invariance of these contents of intuition forms the basis of the constitution of thinking which is determined within itself and is not subject to arbitrariness (see sections 1 & 3). If this fact is not taken seriously then the actual nature of thinking in the form of mathematical intuition cannot be grasped as an imaginative creative process which at the same time occurs totally out of its own necessity.

In contrast to the contents of the process of mathematical intuition, there is the principle, effective and actually active, according to which this process occurs. For, by means of and in accordance with this principle the contents of thought are produced and related one to another.

The state of being of the critically implemented principle is thus something essentially different from the conceptual contents produced with it. The former is actively at work and the latter is passively resistant.

Thus we have found an active and effective principle that does not belong to the world of sensory experience. As there is no *directly* experienceable evidence of such a dependence in this activity, there is no cause to postulate one - unless one wants to state a proper hypothesis. Furthermore, nothing belonging to or taken from the sense world appears in this activity. Everything must first be produced through the activity itself. So far as I am aware, all the evidence for the dependence of such an activity on the psycho-physiological constitution of the human being relates to the consciousness of this activity, not to the activity itself.

This active and effective principle has a property which has so far not been specifically mentioned: It is not itself active but is activated. For, mathematical intuition does not of itself become active within us, but it is we who activate it. In other words: the source of the thinking activity lies not inside but outside the law of thinking. This source of the activity is called the 'I'. Thus the properties of self-activity as well as the activation of other laws must be attributed to the I. In this sense the I as the source of the activity of thinking is a self-activating principle which also has the means to activate other principles (especially the thinking). This points to a principle which is not only effective by itself but also brings about other laws.

7. Spiritual science

The spiritual sciences as university subjects, i.e. the humanities, are concerned with the *products* of the human spirit. Following Goethe and especially his concept of an experience 'of a higher kind' (1792) Rudolf Steiner (1861 - 1925) developed anthroposophical spiritual science.[19] He followed Goethe only in the historical sense and developed a systematic exposition of this science, based on its own foundations, on direct observation and independently formed concepts. Thus, this science was directed above all to the spirit *actually at work*, i.e. to spiritual principles active and effective out of themselves, principles which in addition are active in the world.

The philosophy of the Middle Ages called the contents of mathematical intuition *universalia post rem*, also *universalia in mente*, meaning phenomena of universal laws in the individual human consciousness. From these are to be distinguished the principles at work in the phenomenal world, especially in nature, the *universalia in re*, as well as the principle effective for itself, in itself (and not in another), *universalia ante rem*.

In keeping with the foregoing discussion, with the principle denoted by 'I', we are dealing with a *universale in re*, that is with a principle active in thinking. With this the existence of *one* such principle is demonstrated and thus too the real possibility (and not merely the conceivability in the sense of a hypothesis or ideal possibility) of a science of the spirit in action, and this with a clarity which does not sink below the level of mathematical intuition, but rises above it. For, we are not dealing here with a kind of mystical enlightenment, but with a process which can be carried out with mathematical precision by any individual who wishes to. That there may be other experiences of the same kind with other contents, i.e. experiences in the same clarity of other active beings, can be expected on the basis of these facts, but cannot be forced from them. In any case however, such an experience cannot be excluded at the outset. Rather does it depend on the life and world circumstances - just as with corresponding events in the sense world, where the experience of certain facts, for instance a particular type of animal in Africa, is determined not only by the human being but also by circumstances not within his control.

Rudolf Steiner's founding anthroposophical spiritual science is directly connected with this 'I' experience in the framework of mathematical intuition and makes it a point of departure and criterion of all further spiritual knowledge which penetrates other realms.[20] In this context he regarded mathematics as an appropriate and fundamental preparation for the path of knowledge in anthroposophical spiritual science[21] and rejected all cognitive methods which made do with less than the clarity and strictness of mathematical intuition. Ultimately mathematical intuition is not a matter of *mathematical* contents of concepts but of non-mathematical contents in the form of cognition proceeding according to mathematical intuition. In his 'Philosophy of Freedom' (1894/18) Steiner used the term 'intuition' essentially for that process we have here called mathematical intuition.[22] Thus Steiner realised in the fullest sense and in mathematical clarity the Platonic ideal of intuiting a being, the soul having been prepared by redirecting it through mathematical cognition. In addition he underpinned the whole of spiritual science with a methodological principle which combines Goethe's insistence on mathematical rigour with the training of a higher organ for supersensible perception.

It can be gathered from this presentation that Rudolf Steiner's spiritual science is neither a utopia or an unattainable myth. It is an improper hypothesis, i.e. a reality achievable in principle by everyone who can involve themselves with mathematical intuition.

If man is in essence a spiritual being and it can be shown that he has a direct access to this essence, then this has fundamental consequences which in many respects makes current scientific approaches seem in need of broadening. Mathematics can play an important role as a path to this insight. Perhaps this role will one day be appreciated as the crucial contribution of mathematics to culture.

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Notes

1 c.f. Radbruch (1989)
2 For a traditional perspective on mathematics, which differs substantially from ours, see for instance Grauert (1986)
3 So far as these are present in human thinking they are also referred to as *Concepts*. With this is not meant the words, the symbols, the spoken expressions, but their conceptual *significance* or conceptual content.
4 For an application of this viewpoint to the interpretation of modern developments in molecular biology, especially in genetics, see Heusser (1989).
5 Plato, 'The Republic', 509-511. For a further discussion of Plato's view of mathematics see for instance Ziegler (1992, Ch.II), and Mittelstraß (1985).
6 Goethe (1792). A more thorough documented treatment of Goethe's views indicated in this section can be found in Ziegler (1993). See also Dyck (1956, 1958) and Ziegler (1992, Ch. VI)
7 See for instance Wille (1988) or Mainzer (1988)
8 In order to avoid misunderstandings, we wish to emphasise that what is meant here are the laws of *pure mathematics*. Therefore, we are not dealing with the problem of agreement of mathematical models with a realm of reality lying outside mathematics.
9 Maddy (1990, Ch. 1) gives a brief succinct overview of this problem and the various attempts to solve it in modern mainly Anglo-American philosophy. He includes a comprehensive bibliography.
10 This has been made especially clear by Quine (1951, p 44-5) and (1948, p. 18-19).
11 See Essler (1990).
12 The natural language is the meta-language of all formal or symbolic languages (like programming languages). c.f. Essler (1990).
13 Bieri (1992), for example, deals with the difficult hitherto unsolved problem of tracing the phenomenon of human consciousness to physiological data.
14 This term stems from Quine (1969). See also Maddy (1990, Chaps. 1 & 2)
15 Edmund Husserl opposed a naturalisation of philosophy and psychology, albeit without lasting success. See for instance Husserl (1911).
16 Gödel's own characterisation is made unnecessarily complicated by his developing it by analogy with sense perception. A discussion of various objections and associated attempts at a naturalised solution can be found in Maddy (1990, Sections 1.3 and 2).
17 From this fact can be explained the largely unproblematic understanding within the international mathematical community.
18 Here, a distinction is made here between *invariance* and *invariability* in that whereas the former is absolute, the latter applies to the human being alone.
19 See Steiner (1884-7), (1886/1924) and Ziegler (1993)
20 See in particular Steiner (1894/1918), (1908/18) & (1911)
21 St einer (1904). See also Ziegler (1992) where thoughts merely indicated here are more thoroughly developed and justified. See also Ziegler (1995).
22 See for example Steiner (1894/1988, Ch. V, p.59, Ch. IX, p.103ff & Ch. X, 122).

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Science Group of the Anthroposophical Society in Great Britain

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Science Group of the Anthropsophical Society in Great Britain

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