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Mathematical Thinking: A Cognitive Adventure Between the Ideal and the Real[27]

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Abstract. Based on an analysis of pure and applied mathematical thinking, some aspects of qualitative mathematical thinking are explored. This has consequences for what we might understand by applied mathematics, as well as mathematical thinking as a preparation and accompaniment of the pathway to spiritual-anthroposophical science.

Keywords. mathematical concepts, mathematical method, pure mathematical thinking, qualitative mathematical thinking, applied qualitative mathematics.

Zusammenfassung. Ausgehend von einer Analyse des reinen und mathematischen Denkens und seiner Anwendung auf Naturprozesse werden einige Aspekte eines qualitativen mathematischen Denkens entwickelt. Daraus ergeben sich Konsequenzen für ein erweitertes Verständnis des Begriffs des angewandten mathematischen Denkens und für das mathematische Denken als Vorbereitung und Begleitung anthroposophisch-geisteswissenschaftlicher Forschung.

Schlüsselwörter. mathematische Begriffe, mathematische Methode, reines mathematisches Denken, qualitatives mathematisches Denken, angewandtes qualitatives mathematisches Denken.

Introduction

The structure of reality is complex. Hence we cannot expect that there is a simple and unique relationship between mathematics and the perceptible world. It depends very much upon one's general approach to reality in order to determine particularly what one considers as an important contribution of mathematics towards cognizing reality.

[27]This paper is a thoroughly revised version of a lecture on «Mathematics and its Methodology» given at the Second General Meeting of the Society for the Evolution of Science (SES) in Spring Valley, N.Y., USA, July 7–8, 1990, and which was published in the Newsletter of the Society for the Evolution of Science, 1991, Vol. 7, No. 1, pp. 27–48. Many thoughts which are only sketched here are worked out in full detail and with complete references in [9].

I shall present here several points of view towards mathematics and mathematical thinking in order to analyze the relationship of mathematics to reality (in its broadest sense) within a sufficiently flexible framework. I do not claim to exhaust the matter; however, these viewpoints might be helpful to determine where and how mathematical thinking could prove to be instrumental for understanding the nature of reality and the reality of nature. In addition, they set the stage for further research into the possibilities and limits of mathematical thinking.

Applications of Mathematical Concepts

In the context of traditional natural science, one thinks of mathematical concepts as having some descriptive value towards natural phenomena. This view is based on the observation that *every* object perceivable by the senses has some qualities which can be expressed in mathematical terms (symmetry, shape, extension, etc.). The question is whether these qualities are essential to the nature of this object or not. Let us call a property *quantitative* (or accidental in the Aristotelian sense) if it can increase or decrease without affecting the essence of the object in question. For example, the exact measure of the surface area of a table or the number of leaves of a tree is not something we consider as characteristic for these objects. Thus we may realize that many mathematical properties of the objects within the sense-perceptible world are quantitative in this respect.

On the other hand we may argue that there are mathematical properties which can be used to determine the exact nature of an object compared to others. For example, the symmetry, and the number of petals and anthers in a flower serve as important indicators of the species to which a plant belongs. There is no simple «more or less» with respect to these symmetries or these numbers; and yet we may hesitate to call them essential properties of the plant as such, even if we are aware of the fact, that they are characteristic features of a flower in a certain stage of development.

From a higher point of view specific mathematical properties loose some of their significance. A structure can be called «functional», if no single manifestation brings to light all of its possible structural relationships. Continued manifestation during certain time periods lead to variations of specific parameters and thus to metamorphosis of forms and substructures. If we think of the archetype of a plant as the dynamic functional structure which «causes» the development from seed to flower, and hence is beyond *all* specific developmental stages, then this essence cannot be linked to any specific mathematical structure or shape. What then becomes important are not structures or shapes as such but *transformations* of them. A plant as such is not determined by mathematical properties; it uses such properties in various stages of its development in order to express its essence in the physical world proper.

Although mathematical properties are much more prominent in the inorganic world, even there they do not encompass the whole. This is certainly the case if one is not content with an understanding of some isolated (or selected) inorganic processes, but strives to understand the whole network of such processes.

Even without this greater quest, mathematical structures or formulas, as such, do not tell us anything substantial concerning reality if they are not interpreted appropriately. An interpretation is not a purely mathematical matter. To be sure, any quantitative analysis which leads to a specific mathematical expression presupposes a qualitative analysis without which the resultant mathematical structure has no meaning. As soon as one expresses these qualities in mathematical terms they appear as being more «precise» yet much more limited. What is needed is not a mathematization of qualitative properties but a qualitative analysis which is as clear and stringent as any mathematical argument (see below).

We conclude that mathematical properties are instrumental for the manifestation of an essence in the physical realm. There is no object whose properties are exclusively mathematical: mathematical concepts are not archetypes of any physical phenomenon, object or process. [5, Goethe und die Mathematik, Chapter XII, pp. 237–241] Furthermore, they do not behave like an agent that manifests itself; they accompany physical manifestation.

From this perspective an appropriate approach to the study of mathematical properties of physical objects, in particular objects of the organic realm, might emerge. First of all, one needs to understand clearly the essential functional qualities of the objects under consideration. By this we mean the essential laws which govern the typical developmental stages of an organic creature. In the next step one could consider the available mathematical patterns in order to understand how, when and why this object uses these mathematical structures to express itself. This will be of particular importance when one wants to describe an evolutionary or developmental process (on the phylogenetic or the ontogenetic level respectively). Here the ultimate question is not: What mathematical properties are manifest at a particular stage of development or evolution, but rather: How are these structures modified or transformed during the development, at which evolutionary stage do they become prominent or disappear altogether? These questions might lead to a more specific understanding of how a non-physical entity, such as the archetype of a plant, uses mathematical properties in order to manifest itself consistently and continuously in the physical world.

Pure Mathematical Thinking

Once discovered, mathematical properties of physical objects can be related *to each other* instead of relating them back to the object from which they emerged. Historically, this step was crucial in the development of mathematics from a natural science into an intellectual adventure for its own sake. [7]

In the tradition of Plato, mathematical thinking became instrumental in freeing the human consciousness from the limitations of the sense perceptible world. Mathematical concepts, as such, are free from these limitations; continued occupation with them prepares «inner» organs to the awareness of non-physical experiences, namely the inner workings of nature and/or man.

Plato taught his students to think about the conceptual structure of a circle and its various representations. In contrast to any specific representation, the pure concept of a circle involves only relationships, not any kind of concrete specification of its elements. Being aware of this distinction means that one knows at any instant of time the distinction between the ever changing things of the physical realm and the eternal nature of the conceptual world. Eventually what counts is not so much the specific concepts one uses in carrying out the exercises Plato suggested, but the thought-activity itself which goes back and forth between conceptual insight and mental image («Vorstellung»). Eventually this activity transcends both these preliminary stages.

One might go one step further and consider not only the relation of a given definition of a circle to a mental image of it, but look at the relation between different definitions of the circle. For example, let us compare the two following definitions of a circle within a plane (Table 1):

A circle is the locus of points which have the same distance from a fixed point, the center.	A circle is the locus of intersection points of the perpendiculars from a fixed point onto the lines of a pencil through another fixed point.

Table 1: Definitions of a circle

In proving the equivalence of these two definitions, the fact that they imply each other one needs to go beyond either one of these definitions in order to grasp that they are only representations of one and the same thing: the circle *per se* (for details, see [11]). These considerations lead one into the realm of *pure thinking*, where one is concerned solely with universal relations instead of individual specifications. This means that one has left the limitations of the sense-perceptible world as well as the compulsory forces within one's personal ego. This is, in effect, the case if and only if the sole guidance is provided by these concepts themselves. However, note that the concepts do not *force* themselves upon us. They only appear when we are actively «looking» for them and at them.

Modern mathematics, with its discovery of infinitesimal calculus, non-Euclidean geometry, abstract algebra and axiomatics, created ample opportunities to delve into the realm of pure thinking. [7] However, one does not need to go as far as modern mathematics to experience pure thinking. A careful analysis of elementary mathematical concepts shows that the capacity of pure thinking can be developed starting with the very basic concepts themselves (namely circle, line, point, number, elementary arithmetic etc.).

Is the training of the faculty of pure thinking the end or the ultimate goal of the educational mission of mathematics? The answer is a definite no. In the next section, several aspects are explored which suggest this conclusion. To be sure, this does not mean that the power of mathematics for developing the capacity of pure thinking should be underestimated. On the contrary, this is a very important goal, but it is wonderful to discover that mathematics can lead further still.

Mathematical Method

Since Descartes and Spinoza the method of mathematical thinking served as a model of scientific thinking in general. Even Goethe claimed to have applied it in his writings on optics and color theory. [8]

To begin with, it is not very clear what exactly is meant by the «mathematical method». One can distinguish at least two different interpretations of this term. In general, applying the method of mathematics means that one aims at a clarity and consistency which otherwise can only be found within the field of mathematical objects themselves. Insofar as no specific mathematical thought-patterns are involved, one might call this method simply logical. The criteria of correct reasoning in this realm are the laws of classical (Aristotelian) logic and the clarity that comes with them.

In particular, the method of mathematics might be considered as coinciding with the method of the inorganic realm. Mathematical reasoning is traditionally based upon a set of axioms (or postulates) which are supposed to be clear and self-evident. The various theorems of any mathematical subject are proved by arguments based on definitions and immediate consequences drawn from the relevant set of axioms. Within mathematics, however, the nature of these axioms cannot be analyzed except to ask about their completeness and logical consistency: they have to be assumed as true. On the other hand, the main task in the physical sciences is to find the elementary or archetypal phenomena which form the key for understanding complex phenomena and processes. [1] Eventually the complex processes of the inorganic realm need to be understood in terms of several archetypal phenomena interacting with each other in various ways.

The *method* of mathematics and that of the inorganic realm thus coincide if we agree to identify axioms and archetypal phenomena. This does not mean, however, that we consider these two fields as identical. There remain significant distinctions. Consider for example the axioms of incidence of plane projective geometry (Table 2):

Two points have exactly one line in common which passes through both points	Two lines have exactly one point in common which lies on both lines.
A line carries infinitely many points.	A point carries infinitely many lines.

Table 2: Axioms of incidence of plane projective geometry

From a purely mathematical point of view,,these axioms comprise a minimal set of necessary and sufficient conditions for an incidence relationship between the elements «point» and «line». However, considering only these axioms, we would not know, what the elements «point» or «line» look like. Several interpretations (or models as the mathematicians say) are possible. Apart from the usual depiction of points and lines on the blackboard or drawing paper, we might consider the so-called spherical model: «Lines» are represented as *great circles* and «points» as *pairs of polar (diametral) points*. As one sees after some thought, these «points» and «lines» satisfy the axioms — but certainly do not *look* like «normal» points and lines. An even more abstract model is the one which forms the basis of plane analytic geometry, where «points» are represented by ordered pairs of real numbers and «lines» by linear equations. From a purely mathematical point of view, there is no indication as to which of these models is the «correct» one.

I conclude that the axioms determine their elements only up to a structural level, which can be satisfied in various ways. There is no full, complete account of the elements (points, lines) as such; the elements remain elusive. On contrast, archetypal phenomena are supposed to account for their elements in a definite, not only in a relative sense. After the relevant set of archetypal phenomena has been found, there should be no incompleteness left as to the true nature of the elements involved.

Taking a closer look at mathematical thinking, we may find two limitations inherent in it. One comes from the observation we just made, namely that mathematical concepts (as components of a structure) are, in principle, incomplete concerning the very nature of the elementary objects they describe (which are elements of a specific model). One might call this the *limitation of content.*

The other limitation is more hidden and lies at the heart of mathematical thinking itself. Mathematical thinking is abstract on two accounts. First it deals with concepts from which all pointers to physical reality have been removed (sense-free concepts). Second, although mathematical concepts are self-sustained and objective, they are abstract in the sense of *passive or powerless* as if all their spiritual reality has been squeezed out of them. [3, Von der Abstraktheit der Begriffe, Chapter IV.3, pp. 138–140] Even though they seem to be more conspicuous than other, say ethical concepts, they do not force themselves upon us — we need to grasp them by activating our thinking. Their passivity guarantees our clarity of insight which depends on the intensity of our spiritual activity within the thinking process. This second limitation might be called the *limitation of form.* It is this limitation that mathematical thinking has in common with pure thinking in general.

How can these limitations be overcome without disposing of the clarity and straightforwardness of the mathematical mode of thinking? The limitation of content might be superseded by studying and developing non-mathematical conceptual categories in the same clear and crisp spirit as mathematical concepts in order to explore more deeply the complex conceptual background of the various natural phenomena. An historical example of this approach can be found, for example, in G. W. F. Hegel's «Wissenschaft der Logik» or his natural philosophy within the «Enzyklopädie der philosophischen Wissenschaften».

To overcome the *limitation of form* poses more of a challenge. Eventually it consists of nothing else than the discovery of the *power of mathematics for spiritual development.* [4, Mathematik und Okkultismus, pp. 7–18] This is a large field in need of more discussion and development. In the following two sections I will discuss two approaches towards this general goal.

Qualitative Mathematics

Goethe summarized his approach to natural science with the concept of *rational empiricism.* [8] This means that one carefully observes the phenomena and processes and subjects them to a conceptual analysis from as many sides as appropriate. There is no preferred viewpoint to begin with: one has to go back and forth between perception and conception in order to find valuable perspectives. The activity, or the organ, which carries out this method was called «anschauende Urteilskraft» by Goethe, something like «imaginative power of reasoning.»

Plato also stressed the importance of working through different levels of mental representations. He thought that this is the best preparation for any serious exposure to the spiritual world. In the following, I present an approach to geometrical thinking which might be thought of as an application of Plato's and Goethe's view to the field of mathematics. The corresponding activity will be called *qualitative mathematical thinking*. This concept was first introduced by R. Steiner (see for example his lecture from April 5th, 1921 [2]).

Let us consider a simple example, the circle. We refer to the distinction we made earlier, namely between the concept and its mental image («Vorstellung»). Taken for itself, the mental image is something rigid with a fixed form. As soon as we vary the specifications of the elements, that is, the location of the plane or center, or the length of the radius, we leave the realm of the fixed forms and enter into the realm of form flows or movements.

To be more specific, consider the inward and outward flow of circles with a fixed center in a fixed plane but with a varying radius (figure 4). As soon as we settle for a certain quality and direction of this movement, the whole picture changes and the center appears as a source or sink respectively. What regulates this flow seems to be the concept of the circle. But it becomes increasingly clear that we need to take into account other concepts as well in order to understand the subtleties of this flow more closely. First, there are the specific orbits or paths that are assumed to be used by the moving points, namely straight lines or spirals through the center, for example. Then there is the problem where the points on the circumference of the outward flowing circles are going. As it turns out, they will merge into the line at infinity. This line taken as a series of points seems to function as a source or sink when the center functions as a sink or source respectively.

Even if one takes into account the line at infinity as a fixed element in this inward and outward flow of points, one realizes that the center-point in fact is responsible for the main features of this flow. All points are moving on lines which pass through the center-point (instead of moving on Archimedean or logarithmic spirals, for example). One might capture the qualitative aspects of this flow if one imagines oneself in the center of a horizontal plane, letting the points flow in and out.

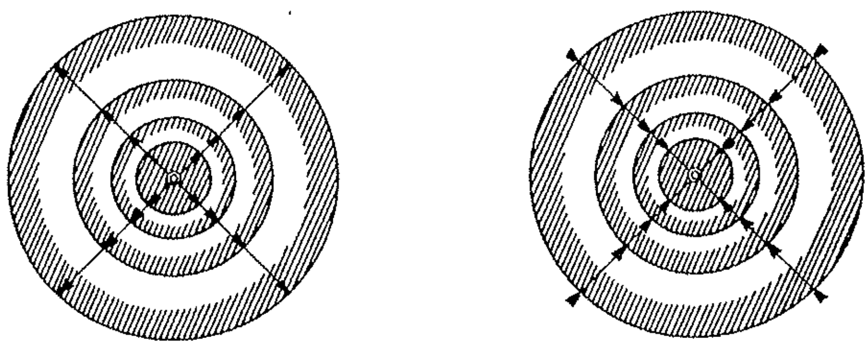

Figure 4: Outward and inward streaming point flow with respect to the center-point

A very different picture appears when one takes into account the tangent lines of these circles instead of the points on the periphery (figure 5). Although the underlying structure is the same, the whole process of circles moving inward and outward shifts its «center» from the mid-point to the line at infinity. Each line is now moving such that it stays parallel to itself, that is, the lines rotate around their points at infinity. Instead of

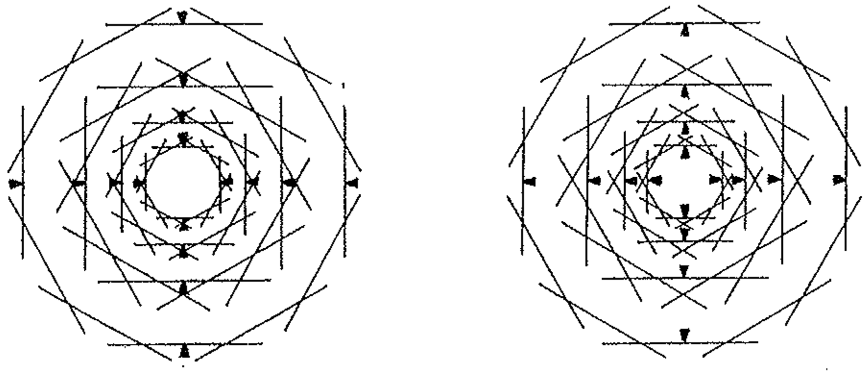

Figure 5: Outward and inward reaching line-flow with respect to the peripheral line (line at infinity)

having points flowing outwards or inwards with respect to the mid-point, we have now lines (tangents) reaching outward or inward with respect to the line at infinity as their source or sink respectively.

To be sure, this is not a complete account of what happens with the points and lines and their interaction during the flow-process described above. Much more could be said. For example, one could look at the outward flow of points (figure 4, left) and the outward reaching lines (figure 5, left) as similar gestures, namely as movements of the elements points and lines away from their respective origin or source. Conversely, the inward flowing points approach their center (figure 4, right) in the same way as the inward reaching lines approach their periphery or line at infinity (figure 5, right). However, this is not the place to go into all these details. Our description is supposed to serve as an example for a general approach which shall be characterized in the following.

A great advantage of (elementary) mathematics in contrast to natural sciences is that it is relatively easy to grasp the relevant conceptual structures in full detail. This puts one in the position to study carefully the relation of these concepts to their mental images. In particular, if one takes into account possible variations of a given geometrical entity, one trains the inner eye to observe the relationship of a functional structure to its varying (transforming, developing, evolving) representations or manifestations. Since we are doing all this, we are in command of everything that happens. Although we are not the one's that create the corresponding conceptual entities, we are certainly involved in the way these concepts manifest themselves as mental images. It is our creative exact phantasy which causes a concept to live through its form variations within the realm of mental images.

Eventually, what comes out of this is a capacity, namely the «anschauende Urteilskraft» (Goethe), which is now prepared by such geometrical exercises to go back and forth between the conceptual and the perceptual level in order to produce a cognitive unity of similar characteristics as the unity between the concept and these self-produced mental image.

It is this close connection between conceptual insight and clarity together with the variations of the corresponding mental images which we call *qualitative mathematical thinking*. A precondition to handle it properly is the capacity of pure thinking. Only then will the whole process be guided by clear conception, and not any kind of «wild» phantasy. On the other side, it is important that our thinking learns to freely individualize its pure insights into concrete manifestations.

From this approach to qualitative mathematics emerges a different conception of applied mathematics than the one by applying straightforward maybe complex mathematical models. We could call it *applied qualitative mathematics*. At first one trains the inner eye through qualitative mathematical thinking as described above (apart from the circle or similar figures, the Cassini curves are very helpful. [9, Chapter 11]) After one has done this for a while, a new capacity has emerged. In the next step one needs to forget the mathematical concept which served as a training instrument; then one applies the capacity (not the concept) to the analysis of observations of natural phenomena.

To be sure, before one can delve into the field of applied qualitative mathematics, one needs to work out in more detail and clarity the nature and content of qualitative mathematical thinking. [9] There have been other efforts in this direction, namely by G. ADAMS, A. BERNHARD, L. EDWARDS, U. HANSEN, G. KOWOL, L. LOCHER, E. SCHUBERTH, O. WHICHER, G. UNGER and others (see the references in: http://mas.goetheanum.org/1672.html and http://mas.goetheanum.org/1671.html).

From Mathematics to Spiritual Science

In this section we shall take up once more the geometrical example from the last section and study it from a different angle. This time, we want to concentrate in particular on the activity which is needed in order to perform this exercise.

In producing the movement of circles going inward or outward as described in the last section, we become aware of the fact that it is our activity which makes the whole process go. The circles do not move of themselves, *we* make them move. On the other side, the rule according to which this movement is structured stays the same all the time: it does not change. If we want to bring this rule itself into our consciousness, we must exert an even greater activity, since this means that we must forget all that relates this rule, or concept, to the sense-perceptible realm. The rule then becomes a *pure* concept. What is relevant here are not any concrete specifications of the elements of this concept (plane, center, distance), but the structure which relates them to each other independent of any individual instance.

Thinking the pure concept of the circle appears as a multi-facetted experience. First there is the clarity of insight into the mutual relationships between the relevant elements. Second, there is the observation of a necessity inherent to these relationships: they cannot be altered or transformed in any arbitrary way. It is of crucial importance to be aware of the fact that the first two observations would not occur in any definite sense if we were not totally involved in our thinking activity. The concept itself appears then as totally passive, it does not force itself upon us. The clarity of the thinking experience depends on exactly this contrast between our own activity and the passivity (or paralyzed state, [3, Von der Abstraktheit der Begriffe, Chapter IV.3, pp. 138–140]) of the concept.

A pure concept lives only through our thinking activity, in our mind. It does not appear by itself. However, it is a something, a universal thought, which manifests itself through thinking. Here one might ask if the *pure*, or sense-free concept is the only mode of appearance of a universal thought in general. Is it necessarily so that a thought content, i.e. the pure concept itself, appears separated from its own activity? In the case of pure thinking, we seem to provide the activity, but this is definitely not the activity of the thought we are thinking about.

But there is an activity which we are well aware of: the thinking activity itself. If we could work out the conceptual structure of thinking as such then we would be actively immersed in thinking a thought which at the same time appears as active itself. This is a major step beyond thinking pure concepts alone: one is now also aware of the thinking activity itself, together with its structure. (What this structure looks like cannot be developed here: we refer the interested reader to [6], see also [10]).

Further observation reveals that it cannot be thinking itself which appears as an activity. If this were the case, it would contradict the experience we described earlier, namely that the thinking process is not something which imposes itself upon us or forces us to do something, but is totally dependent on our intention to carry it out. Hence the activation of thinking is itself effected by something beyond it, namely the very center of ourself, the I (in contrast to the ego-self). There are two possibilities to consider here; only direct observation can decide which one is true. Either the I is a self-effecting agent or it is activated by something else. Only if the first case turns out to be true could we consider our I as something which constitutes the very center of our existence. Otherwise we would be puppets held by strings pulled by some «Super-I».

Since the thinking activity is the only direct experience of an active will force with respect to the normal consciousness, the observation of the spiritual activity of our I can take place only within the thinking process itself. The I is not dependant on thinking, but its manifestation in the normal consciousness is. Once we brought ourself to experience and understand this, it becomes the reference point for every further spiritual insight to come. Because it is the only observation of a universal thought which is at the same time as clear as a mathematical thought, active and its own efficient cause.

All this is well beyond the mathematical thoughts we started from. But it should be clear by now, that a deeper understanding of mathematics leads inevitably to a search for such an experience. The trouble is that further insights and thoughts are needed to develop this line of spiritual development in more detail and in all the subtleties it involves. This cannot be done here. We refer the reader in particular to [6], [3, Von der Abstraktheit der Begriffe, Chapter IV.3, pp. 138–140], [4, Philosophie und Anthroposophie, pp. 66–110] and for a more detailed exposition [9], [10].

In the essay [4, Philosophie und Anthroposophie, pp. 66–110], Steiner develops the conceptual and experiential structure of the I within the framework of the battle between nominalists and realists in medieval scholaticism. He develops in a very concise way the theory of universals and applies it to the I bringing together Aristotle's and Fichte's approach. In the end, he maintains that the resulting knowledge about the working of the I within the thinking process is the basis for any kind of spiritual research which claims to be scientific. Through the experience of the I in this sense shines something into our normal consciousness which otherwise is purely spiritual: it is the only true intuition (in the technical sense R. Steiner uses this word) accessible in terms of our normal cognitive capacities.

Conclusion

Applied mathematics in its traditional sense includes a quantitative analysis of the relevant phenomena and processes as well as the construction of a mathematical model by which forecasts can be made and compared with the available data.

Apart from this direct application of mathematical concepts and the analysis of their fitness one might conceive a different kind of applied mathematics which has been called *qualitative mathematical thinking*.

Since Plato, it is well-known that mathematical thinking is instrumental in developing the capacity of *pure thinking*. To be sure, this is a very important goal in itself. However, this might lead eventually far away from the concrete phenomena itself. The question is whether or not mathematical thinking is of any use for understanding and experiencing reality beyond quantitative analysis or purely methodological considerations.

I propose two different, yet deeply connected, approaches which open up pathways to overcome the inherent limitations of mathematics. One is to develop qualitative modes of mathematical thinking by uniting the conceptual and imaginative approach. The other approach recognizes the fact that mathematical thinking is an instance of pure thinking, and hence might serve as an experimental field to observe the workings of the spiritual activity within our thinking.

Spiritual essence in its deepest sense does not lie within the realm of pure mathematics, but this essence can be approached through the development of qualitative mathematical thinking and therewith we may become aware of the subtle ways our own spiritual essence manifests itself. Once we grasp the inner workings of our own essence, we might be able to understand and become aware of other essences as well.

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