

Geometric Crystal Morphology on a Projective Basis

Towards the Complementarity of Morphology and Structure Theory

Renatus Ziegler¹

1. Introduction

Many people experience an inner delight when beholding regularly shaped crystals, a quiet reverence for their wonderfully regular forms. And thus the question soon arises regarding the nature of crystals, the lawfulness underlying their shapes and properties. In textbooks, popular writings and in museum exhibitions, mineralogists and crystallographers mostly emphasize the lattice structure of crystalline matter, which is said to underlie the vast variety of crystal phenomena. Yet a certain disappointment may be felt as a result of being unable to bring one's inner experience into any sort of deeper relationship with these lattice structures. This essay will draw attention to the fact that the assumption of a lattice structure only expresses *one* aspect of a crystal's nature, and consequently that it can and must be embedded in a more comprehensive relationship.

Phenomenological and experimental investigations of crystals reveal that the majority of their physical and geometrical properties are dependent on entirely distinct spatial directions, immanent in the individual crystals themselves. For example, the level of hardness and the formation of planar cleavages are not the same in all directions; they have maxima and minima in different directions, thus laying the foundation for an initial (inner) orientation of the crystal. Similar properties are exhibited in the thermal expansion, the electrical conductivity, the piezoelectricity, the elastic vibrations, as well as in the refraction of light and polarization phenomena of non-cubic crystallizing minerals.

Here the so-called Neumann principle holds: the symmetries of the physical properties of a crystal at least contain the geometrical symmetries of the corresponding crystal polyhedron. This means that a geometrical examination of the shapes and symmetries of a crystal within the context of geometric crystal morphology furnishes the higher order symmetries that underlie all the symmetries of the specific physical properties. Hence symmetries (especially axes and planes of symmetry) are naturally occurring properties, which may indeed be discovered and ascertained in a specific, finite crystal polyhedron, yet point beyond the boundaries of the individual crystal. For on the one hand the same symmetries may be attributed to different crystals, while on the other the axes and planes of symmetry point beyond the finite crystal body to the entire surrounding space. This opens up the conceptual possibility of no longer simply comprehending crystals in a local-additive

¹ This paper first appeared in German in the *Elemente der Naturwissenschaft* 71 (2), 1999, pp. 1-45. The original German text – “Geometrische Kristallmorphologie auf projektive Grundlage. Zur Komplementarität von Morphologie und Strukturtheorie” – can also be found on the author's website, Renatus Ziegler: Reines Denken (see link: <http://www.reinesdenken.ch>). This text was translated into English by David W. Wood, and the translation was checked by the author and his suggested emendations included. It appeared in *Archetype, the Journal of the Science Group of the Anthroposophical Society in Great Britain*, ed. Dr. David. J. Heaf, no. 6, September 2000, pp. 1-45.

manner, as structures solely arising from the reciprocal effects of elementary particles. Therefore crystals may also be understood as distinctly unified forms, having their origin in non-localized formative principles, which, considered geometrically, correspond to configurations spanning the whole of space.

Among other things the present investigation is a case study of a central problem – how to undertake an anthroposophical/spiritual-scientific extension of traditional natural-scientific knowledge. Using crystallography, it will be shown how out of a “sound instinct for knowledge” (*Steiner* (1918), p. 107), crystal physicists have concentrated on the inner, objectively conditioned necessity of one of the fundamental assumptions of scientific crystallography. This at once paves the way for an extension into the conceptual possibility of spiritual forces – yet without coming into conflict with any of the fundamental laws of physics. This fundamental assumption concerns the infinite nature of a crystal lattice (see *section 6*). It also opens the door to so-called reciprocal space, which plays such a vital role in solid-state physics (*section 7*).

To begin with, the present study provides a summary of the most important morphological viewpoints (*section 2*) and the classic crystallographic laws (*section 3*), as well as outlining the systematic construction of the theory of crystal forms (*section 4*). *Section 5* on morphology and structure theory examines the two complementary interpretations of the fundamental law of crystallography, the lattice structure hypothesis, and the crystal morphological hypothesis. The relationship between the two will be discussed in greater detail in *section 6*. This latter hypothesis will require some projective geometry, which will be considered in *section 5*. The article concludes with a brief look at a number of physical considerations (*section 7*), as well as the structure and morphogenesis of crystals and their synthesis (*section 8*).

2. Geometric Crystal Morphology

Geometric crystallography or crystal geometry is not concerned with real crystals and their abundance of specific morphological features, but only with idealized forms. In particular we will take our start from the fact that crystals are spatial bodies, bounded on *all sides* by *plane* faces and furnished with clearly defined *straight* edges and vertices, called *crystal polyhedra*. All in all the following study will confine itself to single crystals: the discussion will neither include twinned and multiple formations, nor crystal aggregates.

As *box 1* shows, this restriction still allows ample scope for the morphological diversity of crystals of a single type. Nevertheless, the five different specimens of quartz crystal and the three different specimens of fluorite crystal are only distinguished by the highly varied development of their faces. We discover that every face of one of these specimens has a corresponding face situated parallel to it; that is, it has a face in the same ‘position’ in every other individual crystal. With respect to the specific symmetries, the ‘equivalent’ faces in a quartz specimen are denoted using the same letters.

Crystal polyhedra which are in a certain sense geometrically related are ordered into one and the same *crystal system*. The relative *positions of the faces* are alone significant for the properties of a crystal in a crystal system. A characteristic feature of crystal polyhedra generally is the lack of a uniquely defined centre. Among other things this also reveals that the location of the crystal nucleus in the beginnings of crystallization can no longer be generally ascertained in the outward growing crystal. Thus for geometric crystallography, in addition to this *first* idealization – limiting ourselves to crystal *polyhedra* – a *second* is now added: only the *relative positions of the faces* shall be considered, independent of their spatial location.

A careful study shows that *symmetries* are the most essential principles of any geometric shape, and characteristic for a crystal system; yet they only come to expression in

the external shape in a concealed manner. If the external shape is a perfect expression of the corresponding symmetries then we may speak of ‘archetypal forms’; in crystallography they are termed *simple forms* or *ideal forms*.

Box 1: Morphology of ‘Distortion’, Law of Constancy of Angles

It follows from the comparative observation of different crystals, as well as from crystal growth and phantom-crystals, that in the corresponding spatial ordering of crystal specimens there exist crystal polyhedra of such a kind which give rise to genuine pairs of parallel faces (apart from the completely non-symmetrical triclinic class). These pairs of parallel faces belonging to the different crystal specimens are called *corresponding faces*; we say that such faces have the same position. Corresponding crystal specimens are ordered into the same *crystal system*. With regard to the distinct crystal systems in crystal geometry, only the *relative positions* of the faces of the different crystals are taken into consideration (i.e. neither the face’s size, exact location in space or boundaries are relevant). The foregoing is a consequence of the *law of constancy of angles*.

Law of Constancy of Angles: *The angles between the faces of different crystal polyhedra belonging to the same crystal system are identical under constant conditions.*

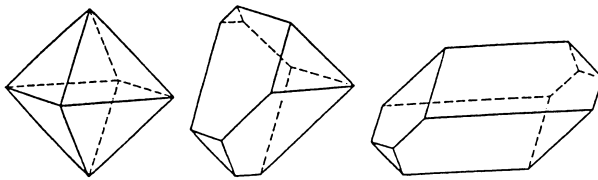


Figure 1: Variations or ‘distortions’ of a regular octahedron (from Kleber (1990), Figure 1.10, p. 26)

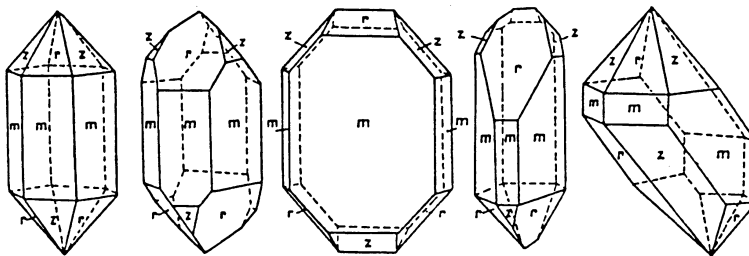


Figure 2a (above): Variations or ‘distortions’ of a trigonal crystal form (from Niggli (1949), Figure 1, p. 54)

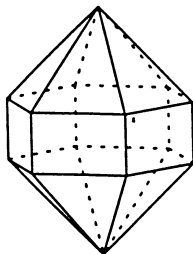


Figure 2b (left): Tangential polyhedron of the crystal forms depicted in figure 2a

The following investigations are only concerned with *general crystal morphology*, that is, with the principles underlying all crystal polyhedra in each crystal system, and not with the specialized morphology of actual crystals. Hence, neither the conditions for crystal growth and formation (see *section 8*), nor the constituents of their outer-surfaces, nor non-polyhedral crystals (e.g. such as dendrites) will be in any way investigated.

The fundamental laws of general crystal morphology will be presented in the following *boxes 2-5* and in the relevant sections. They form the basis for the more detailed

and systematic examinations into the complementarity of morphology and structure theory outlined in sections 5 & 6.

An extensive treatment of all the themes and problems discussed in this paper may be found in *Ziegler* (1998), which also contains a more comprehensive list of specialized literature. Nonetheless, the fundamental studies on the complementarity of crystal morphology and structure theory have been revised, supplemented and more precisely formulated for this paper.

3. Crystallographic Laws

Through the comparative observation and measurement of the external surfaces of crystals two groups of laws have been discovered which lie at the very heart of classic crystallography.

Fundamental laws of crystallography: Law of constancy of angles (*box 1*), law of rational indices (*box 2*), law of zones and the zonal relationship law (*box 3*).

Crystallographic symmetry laws: General law of crystal symmetry (*box 4*), crystallographic restriction (*box 4*), law of crystal classes (*box 4*), singular symmetry directions: crystal systems (*box 5*).

Fundamental Laws of Crystallography

It is possible to show that the *zonal relationship law* is equivalent to the law of rational indices.

The law of constancy of angles is a special case and therefore follows from the zonal relationship law. The law of constancy of angles is connected with the arrangement of the faces of different crystals, which are of the same type as the crystal polyhedra of a crystal system; while the law of rational indices and the zonal relationship law relate to every possible type of arrangement for the faces of crystal polyhedra belonging to any crystal system.

In the final instance only one fundamental law remains, which we may designate as the *fundamental law of crystallography*. Mathematically speaking, it is immaterial whether we choose this to be the law of rational indices or the zonal relationship law. From the morphological standpoint more value is placed upon the zonal relationship law.

Tangential Polyhedra

It especially follows from the law of constancy of angles that our morphological-geometrical investigations needn't concern the actual shape of the faces of crystal polyhedra, but that we only need to acquire an overview of the ratios between the angles and positions of the respective faces. Hence the *tangential representation* of a crystal polyhedron may be of use, in which the geometrically relevant elements are moved toward one another in a parallel manner. Here only the spatial 'position' remains fixed, but this is neither true of the exact spatial location nor of the face's shape: if we choose any sphere within a crystal polyhedron and move all the planes of the faces individually through parallel translation until they touch the sphere and allow the planes to intersect, we then obtain a polyhedron which may be termed the *tangential polyhedron* or the *tangential representation* of a general crystal polyhedron. The centre of the sphere is also the *centre* of the tangential polyhedron. Naturally, as a result of this construction every tangential polyhedron has an inner sphere and all the distances between the faces and the centre (central-distances) are of equal size, i.e. all the faces are *equidistant*. Hence the exact size of a tangential polyhedron ceases to be relevant.

Box 2: Law of Rational Indices.

Three faces α , β , γ of a crystal polyhedron, which have a point of intersection in common, jointly determine together with the common point of the three converging edges A , B , C a general right-angled co-ordinate system. The position of any fourth face ϵ not passing through the common point is then determined by the *ratio of the intercepts on the axes* $a : b : c$ relative to the axes A , B , C (*Weiss index*). Hence parallel faces, that is faces in the same position, receive the same ratios for their axial intercepts.

By means of numerous empirical investigations (goniometric measurements of crystals) it has been established that for a further crystal face position δ , with the ratios of its axial intercepts $a : b : c$, the quotients of the corresponding intercepts

$$a/a' : b/b' : c/c'$$

essentially behave like small integers (including 0). If we choose the intercepts on the axes of the faces to be units and therefore the four faces α , β , γ , ϵ to be the so-called *co-ordinate tetrahedron*, then these parameters which are expanded or shortened respectively to become the integral values h , k , l with

$$h : k : l = a/a' : b/b' : c/c'$$

are called the *Miller indices* for the position of the face δ , written (hkl) , or simply, the *integral indices of the face*. This results in the following:

Law of Rational Indices: *The indices for all the positions of the faces of every possible crystal polyhedron belonging to a distinct crystal system have small integral values, if they can be referred to any four particular faces of one of the crystal polyhedron belonging to this crystal system, but which still constitutes a general (co-ordinate) tetrahedron.*

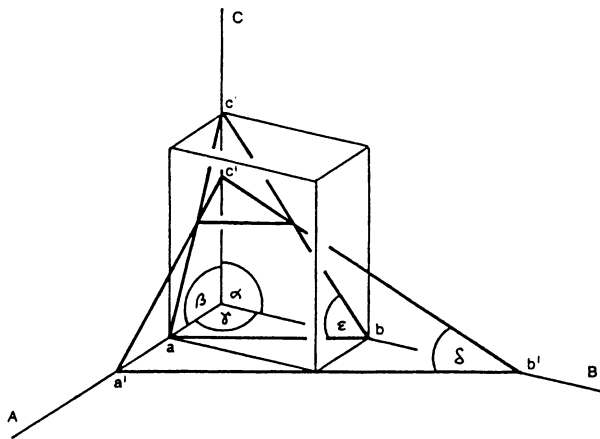


Figure 3: Co-ordinate axes for the indices of the faces with the co-ordinate tetrahedron α , β , γ , ϵ (from Ziegler (1998), figure 3.3, p. 20).

Sometimes crystallographers call the tangential polyhedron of an actual crystal polyhedron an *ideal form*. The crystal shapes depicted in *Box 1* are then termed *distortions* of this ideal form – and which are only fixed with respect to the ratios between the angles of the faces (without all of them having to have the same distance from the centre). The polyhedron on the left in *figure 1* is a tangential polyhedron of the two figures on the right; a tangential polyhedron for the crystal forms in *figure 2a* is presented in *figure 2b*.

Box 3: Zonal Relationship Law (Weiss Zone Law)

A striking property of most crystal polyhedra are their sets of parallel edges (except for the polyhedra of the completely non-symmetrical triclinic class.) We call the totality of the positions of the faces which are parallel in one direction, i.e. parallel to the *zonal direction* or to the *zonal axis*: a *zone of faces* or a *zonal relationship*; the directions of the intersecting edges of the faces in a zone are parallel to one another and to the zonal direction.

A face simultaneously lying in two zones and intersecting the faces of these zones has two pairs of parallel edges as its boundaries. Conversely, if given two zones, we can derive the position of the face that is parallel to the direction of the two edges. Experience with actual crystals shows that this face is a possible crystal face.

This leads us to the question: can we derive from a certain number of positions all possible positions of the faces of crystal polyhedra belonging to a distinct *crystal system*, in such a manner that every new position of a face is determined by two of the zones of the positions which are already present? An answer is supplied by the –

Zonal Relationship Law: *The positions of the faces of every possible crystal polyhedron of a particular crystal system may be derived in a few steps from any four selected faces of a crystal polyhedron in this crystal system which nevertheless form a general tetrahedron: each pair of lines of intersection (zones) of the faces already present determines a new position of a face; if one of these new faces intersects one of the faces already present, we obtain new zones etc.*

Crystallographic Symmetry Laws

Using clearly defined geometrical transformations it is possible to show that the tangential polyhedra of a crystal polyhedron may be allowed to intersect with one another. Transformations of this type are called *symmetry operations*; here rotations about an axis, reflections in planes and through points as well as their combinations all play a role – every one of these operations leaves the centre of the tangential polyhedron invariant. The geometrical objects (points, lines, planes) left invariant by these operations are called *symmetry elements*. The totality of symmetry operations of a tangential polyhedron is called the *symmetry group*, while the totality of symmetry elements is called the *symmetry framework*.

As an example we will describe the largest symmetry group of the cubic crystal system, called by crystallographers the *cubic holohedry*. It possesses the following symmetry elements, which include all the possible symmetries of a cube, regular octahedron, rhombic dodecahedron, and many other solids: 3 fourfold axes (tetrads), 4 threefold axes (triads), 6 twofold axes (diads), 9 planes of symmetry, 1 centre of symmetry (*figure 4*).

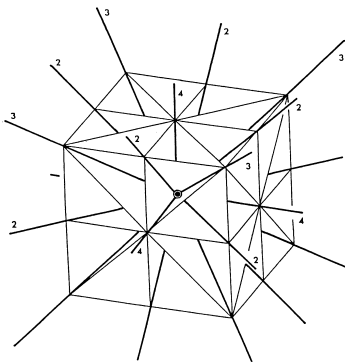


Figure 4. Symmetry framework of the cube (from Ziegler (1998), figure 4.15a, p. 68)

The fundamental law of crystallography and the laws of crystal symmetry cannot be derived from one another; this means that they represent two independent restrictions of the total conceivable diversity of forms. On the one hand the zonal relationship law or the law of

rational indices fails to provide any information about the symmetrical configuration of the possible boundary planes of a crystal polyhedron. Of course, there is no crystal polyhedron that does not satisfy the fundamental law of crystallography. Or expressed in another way: every crystal form satisfying the symmetry laws also satisfies the fundamental law of crystallography. It is precisely the laws of crystal symmetry which secure (from the mathematical standpoint) that there exist crystal polyhedra for *every* crystal class, and not merely highly symmetrical or completely non-symmetrical crystal forms.

On the other hand, the crystal symmetry laws only relate to those faces of a tangential polyhedron of a crystal that are joined with one another through the symmetry law, i.e. through the symmetry operations of a group; that is, they only pertain to the possible faces of one and the same *simple* crystal form. Nevertheless, as a rule crystals consist of combinations of numerous simple crystal forms of this kind. Hence the symmetry law does not say anything about the reciprocal relations for the indices of crystal faces belonging to different simple crystal forms. Yet among those simple forms of faces belonging to one of the 32 crystal classes there also exist limiting cases of polyhedra possessing symmetries not permissible in crystallography (especially axes of multiplicity 5, 8 & 12, see *Box 4*) if the limitation to integral indices is not explicitly demanded. (Example: considered mathematically, in a limiting case pyritohedra could also be regular dodecahedra; however, the fundamental law of crystallography excludes this possibility because it is not consistent with the irrational properties of the golden ratio underlying the regular dodecahedron.)

Box 4: Crystal Symmetry Laws

General Symmetry Law: *The tangential polyhedra of crystal polyhedra are symmetrical with respect to operations that leave the centre of the tangential polyhedron fixed.*

We can show using the law of rational indices, that within the symmetry operations only rotations through angles of $360^\circ/2$, $360^\circ/3$, $360^\circ/4$, and $360^\circ/6$ are compatible, that is, rotations with periods of 2, 3, 4 & 6. This is the so-called

Crystallographic Restriction: *The rotational components of the symmetry operations of tangential polyhedra only have periods of 2, 3, 4, & 6.*

Together with the crystallographic restriction, establishing possible theoretical symmetry operations for the tangential polyhedra of general crystal polyhedra yields 32 different sets or classes of symmetry operations, which are closed with respect to their combination. (These classes are also called crystallographic point groups.) – These are the 32 so-called crystal classes:

Law of Crystal Classes: *The symmetries of the tangential polyhedra of general crystal polyhedra belong to one of 32 crystal classes.*

4. Morphology of Crystal Forms

Morphology of Simple Crystal Forms

The symmetry elements of a crystal polyhedron are characteristic in that they are not confined to the boundaries of this solid, but permeate the whole of space. Together they form an object configuring the entire surrounding space or the ‘surrounding-periphery’ of the crystal polyhedron. Likewise, the corresponding symmetry groups are not limited to this polyhedron. They are uniquely defined mathematically, independent of their relationship with crystal polyhedra.

The 32 possible symmetry groups for the tangential polyhedra of real crystals are summarized in the 32 crystal classes (*box 4*). Each of these well-defined mathematical groups contains a well-defined set of symmetry operations. These operations may then be conversely employed to generate polyhedra – the so-called *simple crystal forms* – in that any tangential plane of a sphere is moved according to all the symmetries of this group (leaving the centre of

the sphere invariant). These planes are then made to intersect, with their common ‘nucleus’ forming the desired crystal form. Other crystal forms are produced, depending on the relative position of the planes to the symmetry framework.

Box 5: Crystal Systems

If we examine the reciprocal relations between the directions of the major symmetry axes, or which amounts to the same thing: the relations of the zonal directions within a zonal relationship, or even the relations between the edges of zonal tetrahedra in connection with the zonal relationship law, then we discover that for every tangential polyhedron of a crystal polyhedron there only exist three significantly different types of symmetry directions, or directions of the axes. These may be classified into seven different systems, the so-called crystal systems:

Law of the Seven Crystal Systems: *The geometrical arrangement of the symmetry directions or the directions of the zone axes for crystal polyhedra may be divided into seven systems. These seven systems of crystal axes are simply called the seven crystal systems.*

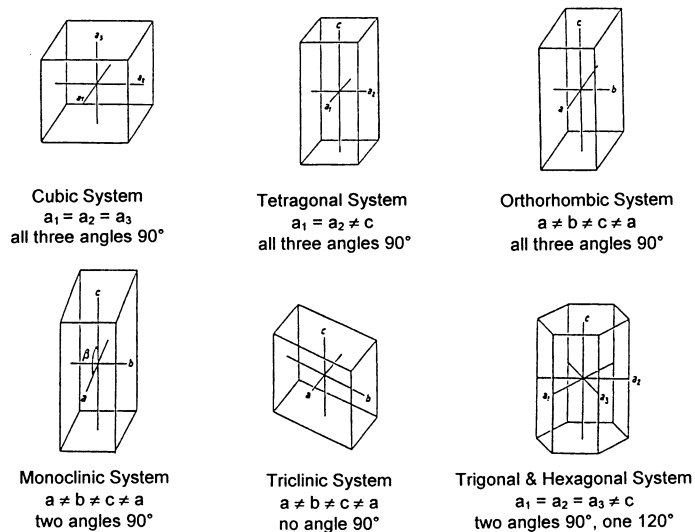


Figure 5: Symmetry and co-ordinate axial directions of the seven crystal systems (from Parker (1975), figure 57, p. 84).

Let us now consider the simple crystal forms belonging to the largest symmetry group of the cubic crystal system, called by crystallographers the *cubic holohedry* (figure 6). Among others this group possesses the following symmetry elements: 3 fourfold axes, 4 threefold axes, 6 twofold axes (see Section 3, figure 4).

If we place a plane perpendicular to a fourfold axis and move it according to all the symmetry operations of the group, there arises a regular hexahedron (cube). And in a corresponding manner there arises a regular octahedron or a rhombic dodecahedron if the so-called *generating plane* lies perpendicular to a threefold or a twofold axis respectively. If the generating plane which lies perpendicular to the fourfold axis is slowly moved in the direction of a twofold axis (where it always remains perpendicular to the plane of symmetry joining the two axes), there arises the metamorphosed forms on the right-hand side of the rounded triangle in figure 6. Corresponding metamorphoses arise in the transition from the twofold to the threefold axis (lower side of the triangle) and from the threefold to the fourfold axis (left side of the triangle). In the middle we have transitional forms in which the generating plane neither lies perpendicular to an axis of symmetry nor to any of the planes of symmetry.

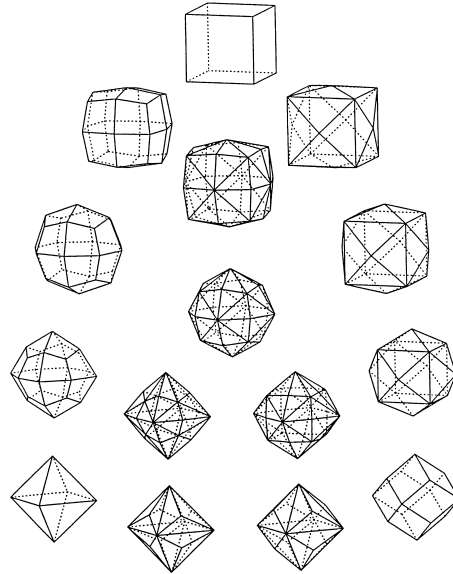


Figure 6: Simple crystal forms of the cubic holohedry (from Ziegler (1998), figure 5.30a, p. 133)

However, with all the groups and positions of the planes there not only exist closed polyhedra as crystal forms; there are also so-called *open forms*. With real crystals they only appear in combination (see below) with other open or closed simple crystal forms. An overview of all open and closed simple crystal forms is given in figure 7.

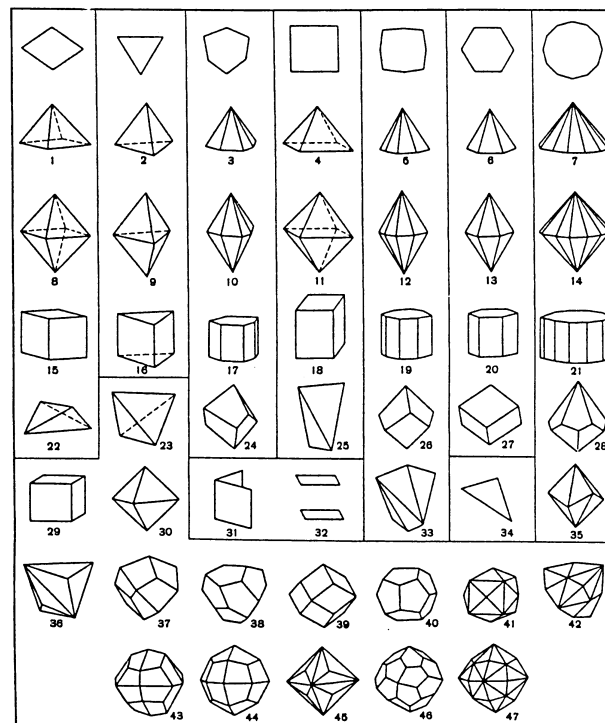


Figure 7: The 47 simple crystal forms (from Shubnikov/Koptsik (1974), Figure 73, p. 74).

Morphology of Combination Forms

Crystal forms belonging to the same crystal system can appear in combination, that is, they form so-called combination forms out of two or more simple crystal forms. In this way the vertices and edges of the one polyhedron become truncated by the faces of another polyhedron or vice-versa.

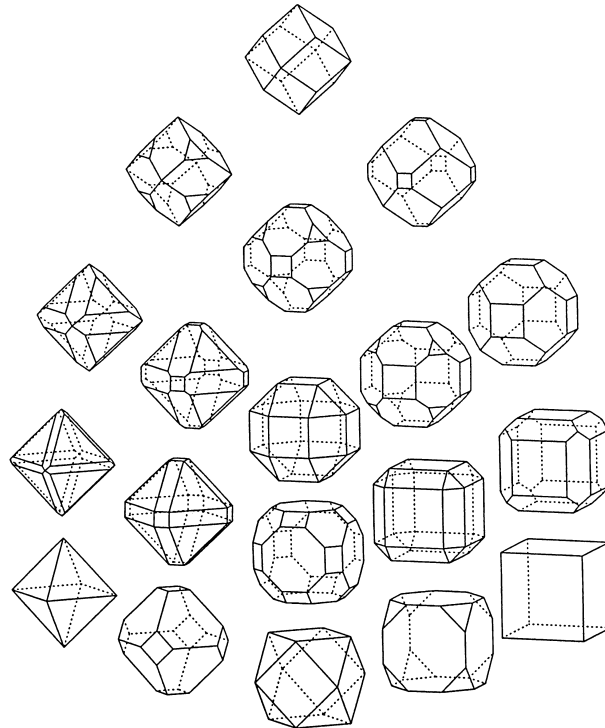


Figure 8: Combinations of the faces of three simple crystal forms belonging to the cubic holohedry
(from Ziegler (1998), fig. 5.46a, p. 151)

Figure 8 illustrates in the form of a so-called combination trigon the combinations of the faces for the polyhedra lying in the vertices of this trigon: the regular octahedron, the hexahedron and the rhombic dodecahedron. The borders of the combination trigon exhibit the combination forms out of the two polyhedra in the neighbouring vertices; the combination forms out of all three polyhedra in the vertices lie in the middle.

The Law of Crystal Forms

Joining the law of crystal symmetry with the fundamental law of crystallography yields a complete overview of every possible crystal polyhedron.

Law of Crystal Forms: *There exist 47 different simple crystal forms. Every convex polyhedron with the following properties is a general crystal polyhedron: Take a finite number of simple crystallographic forms and combine them into a convex polyhedron, then add a parallel displacement of one or more faces ('distortions').*

The crystal polyhedra that arise in this manner are generally no longer symmetrical polyhedra if the parallel displacement of the faces is not an exact similarity. This type of occurrence is termed in crystallography a *distortion of the ideal shape*. Without any loss of generality our present discussion will be mostly confined to the symmetrical, 'undistorted' crystallographic forms of crystal faces.

In crystallography, the simple shapes of the faces that appear in combination in an actual crystal polyhedron are technically designated as its *form*. The *habit* refers to the relative size of the faces, and is greatly dependent on the degree of distortion. The form refers to the ideal shape, the habit – to the actual concrete shape of a crystal.

5. Morphology and Structure Theory

In crystallography we distinguish between two standpoints: the morphological, and the structural. The morphological or macroscopic standpoint views crystals as *continuous* bodies of *finite* size, while in contrast the structural or (sub)-microscopic standpoint views them as *discontinuous aggregates*. In the latter case one proceeds in crystallography and theoretical solid-state physics from the fact that the number of particles (geometrical: points; physical: atoms, ions, molecules) forming the discontinuum is so large in relation to the macroscopic dimension that it can simply be set at *infinity* (see, for example, Kleber (1990), p.16; or Landau/Lifschitz (1979), p. 382, note 1). This will be further discussed below.

One of the crucial aims of this essay is to draw attention to the fact that each of these two standpoints has an inner kernel independent of the other, yet which stand in a complementary relationship and may be united in a mathematically precise manner without either of the two being replaced or rendered obsolete by the other.

Interpretations of the Fundamental Law of Crystallography

Fundamental to the whole of crystallography is the *law of rational indices*, also called the *fundamental law of crystallography* (box 2); a law discovered in the first half of the 19th century on the basis of comprehensive empirical-geometrical investigations (goniometric measurements) of real crystals.

The two standpoints mentioned above lead to different interpretations of this law.

The following hypothesis suggested in contemporary crystallography tends to favour the *structural standpoint*:

Lattice Structure Hypothesis: *Crystal bodies may be understood as if consisting of a finite number of congruent constitutive units (unit cells), which have as their basis an infinite lattice with a translational periodicity in three different directions. These constitutive units in turn consist of a finite number of particles (geometrical: points; physical: atoms, ions, molecules) of finite size and finite distance.*

With regard to any interpretation of the fundamental law of crystallography based on the lattice structure hypothesis and the crystal morphological hypothesis outlined below, I place especial value on the idea of ‘interpretation’ in contrast to ‘explanation’, ‘cause’ or something similar. In particular, our view concentrates on the conceptual *relations* between the concrete phenomena, in this case the geometrical-crystallographic configurations, and not on ‘explanations’, ‘derivations’, ‘deductions’ or ‘reductions’ etc. of the phenomena.

On the basis of the construction postulated here one also speaks of a *modular* or *additive formative principle* existing in crystals. In addition to the *periodicity*, we discover that the *infinitude*, *discreteness* and *homogeneity* of the underlying lattice structure are all decisive; that is, the assumption that crystals consist of a finite number of particles with specific finite distances within a constitutive unit (unit cell), and that are equally distributed according to the infinite lattice structure. The lattice structure hypothesis especially allows us to determine metrical properties such as the lengths of the edges and the angles of the unit cell.

Without any further assumptions, it is evident from the geometrical standpoint that the polyhedral shape of crystalline matter cannot be inferred from its lattice-like structure. As a consequence, it must be stated that, above all, crystals develop planar faces; this is the law of crystal faces.

Law of Crystal Faces: *Crystals develop planar faces.*

As a rule faces don't arbitrarily appear in crystal shapes, but in an entirely distinct manner. This is established by the law of correspondence. This principle which is fundamental to structure theory, implies that a very precise relationship exists between the morphological properties and the lattice structure (*figure 9*).

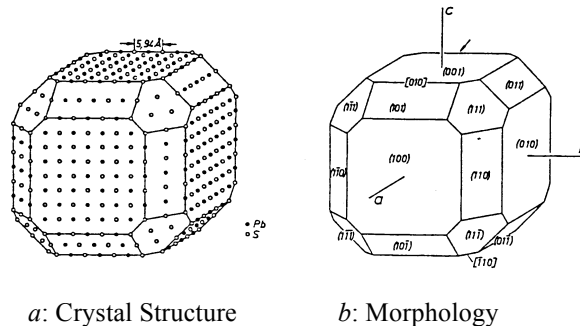


Figure 9: Example of the correspondence between crystal structure and morphology in galena (PbS) (from Borchardt-Ott (1997), p. 30, figure 5.1 a, b)

Law of Correspondence: *In the event that a crystal develops planar faces, these will preferentially appear in lattice planes that are particularly densely packed with particles (or that are parallel to planes with a large packing density).*

The law of correspondence does not say anything about the existence of planar faces in crystals – this is assumed – but concerns the reasons *why certain faces* appear to be dominant.

Together with the law of crystal faces and the correspondence law, the lattice structure hypothesis furnishes a possible interpretation of the *law of rational indices*, which means in effect an interpretation of the fundamental law of crystallography. For lattice planes with small integral indices are not only densely packed planes, but planes which may be rationally constructed from the co-ordinate tetrahedron and hence also from the symmetry framework; – or put another way: whose lines of intersection with the “plane at infinity” belong to the projective representation of the symmetry framework, or may be rationally constructed from it in a small number of steps (see further below). This means that the lattice hypothesis fits in well with the empirical findings expressed in the law of crystal faces, yet without thereby forfeiting its status as a hypothesis. Hence according to impartial phenomenological methods of scientific investigation, it shouldn't dominate the interpretative discussions.

The Crystal Morphological Hypothesis and Projective Geometry

Taken in its strict sense the *morphological standpoint* leads to an altogether different interpretation of the fundamental law of crystallography. This is not entirely obvious, as nowadays the morphology of crystal forms is mostly presented *from the outset* in connection with the lattice hypothesis (see, for example, Vainshtein (1994), Kleber (1990)). Yet a systematic morphology of crystal forms may also be developed without recourse to the lattice structure hypothesis. Thus we must first of all work out an interpretation of the fundamental

law of crystallography that is appropriate for geometric crystal morphology. Only when the domain of geometric crystal morphology is made into a completely independent field will we be able to objectively tackle the problem of how to bring morphology and lattice structure theory together, to form a unified theory of crystal form and crystal structure.

Box 6: Basic Harmonic Configuration

The basic harmonic configuration is one of the most elementary and important figures in projective geometry. The construction of the planar version of this figure starts with four points or straight lines in a general position, i.e. no three of the four points or lines respectively lie on a line or pass through a point.

Complete Quadrangle: There are 6 lines (sides) joining the 4 points A, B, C, D in a general position (vertices: indicated by circles). These intersect 3 further points R, S, T (adjoining vertices: indicated by large filled circles), which are joined with 3 lines (adjoining sides).

The adjoining sides intersect the sides in 6 new points (indicated by small filled circles), three of them lying respectively on one of the 4 (dashed) lines. This yields the *planar basic harmonic configuration*.

Complete Quadrilateral: There are 6 points of intersection (indicated by small filled circles) for 4 (dashed) lines in a general position (sides). These can be joined with 3 further lines (adjoining sides), that intersect the 3 points R, S, T (adjoining vertices: indicated by large filled circles).

The adjoining vertices are joined with the vertices in 6 new lines, three of them passing respectively through one of the 4 points A, B, C, D (indicated by circles). This yields the *planar basic harmonic configuration*.

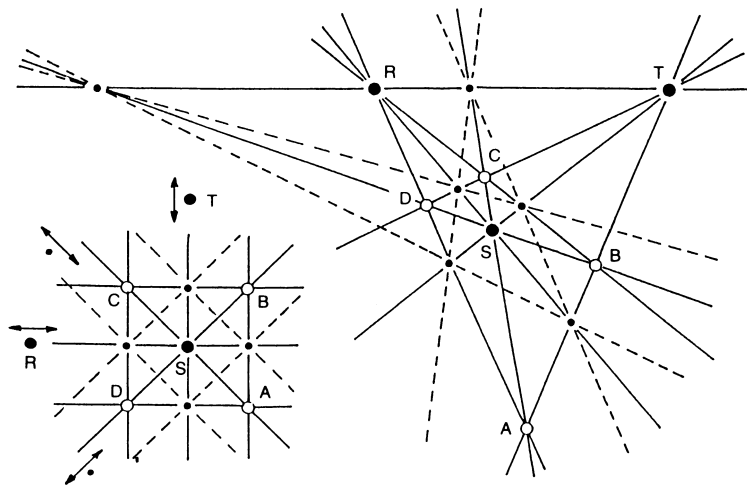


Figure 10: Particular (Euclidean symmetric) and general cases of the basic harmonic configuration (from Ziegler (1998), Figure 6.1a, p. 169)

From the geometrical standpoint, the fundamental law of crystallography is equivalent to saying that every possible position of a crystal face belonging to a crystal polyhedron in a crystal system, can be obtained from the faces forming a tetrahedron by simply intersecting the faces and joining the intersecting lines through further faces. This is the so-called *zonal relationship law* (Box 3).

Because only the positional relations between the points, lines and planes play a role in the zonal relationship law, with no metrical relationships involved at all, this law directly leads into the domain of projective geometry. Moreover, four planes determine a basic harmonic configuration (box 6) in any arbitrary plane, especially in the ‘plane at infinity’ of

three-dimensional Euclidean space. This ‘plane at infinity’ is a geometrical object which behaves like a plane and contains both the ‘point of intersection’ of parallel lines and the ‘lines of intersection’ of parallel planes (cf. Ziegler (1998), section 2.1).

At the beginning of section 4 we described how the symmetry elements of a crystal polyhedron represent a formation configuring the whole of space. Projective geometry offers an objective context in which to understand formations of this kind, since it also includes the geometrical relationships (with respect to Euclidean geometry) in the ‘infinitely-distant’ plane.

For a long time now crystallographers have employed stereographic projections to represent and measure crystal polyhedra. These projections involve nothing more than a distinct rendering of the geometrical properties of crystal polyhedra and crystal symmetries with respect to the ‘plane at infinity’ of Euclidean space.

In order to visualise the concrete relationship between the basic harmonic configuration and the symmetry framework of the entire cubic group, *figure 11* in *box 7* illustrates a ‘perspective’ representation of a cube or a regular hexahedron with its 13 axes of symmetry and their intersection with the ‘plane at infinity’. The intersection of the symmetry framework with the ‘plane at infinity’ essentially produces a basic harmonic configuration. This same idea also underlies the representations in *figures 12* and *13b*.

Box 7: *Symmetry framework of the cube in projective representation*

The basic harmonic configuration in *figure 11* lies in a plane representing the ‘plane at infinity’; it passes from the top in the foreground down towards the back. Its points and lines bear the same notation as *figure 10* in *Box 6*. The 3 fourfold symmetry axes of the cube pass through the adjoining vertices, the 4 threefold axes through the vertices and the 6 twofold axes through the 6 additional points of the basic harmonic configuration. The 9 planes of symmetry intersect the ‘plane at infinity’ in the traverse lines of this figure. The four dashed lines correspond to the mirror-rotational planes that are perpendicular to the threefold axis. These intersect the cube’s outer-surface in regular hexagons.

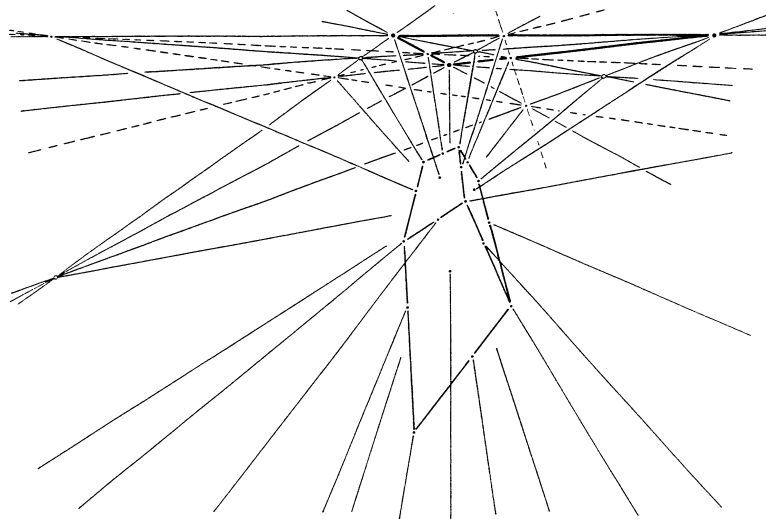


Figure 11: Symmetry axes of the cube or regular hexahedron projectively represented with the basic harmonic configuration in the ‘plane at infinity’ (from Ziegler (1998), figure 6.8, p. 176).

This allows us to deduce the following: if a basic harmonic configuration defined by the zonal relationship law is joined with the symmetries of Euclidean space, then the more or less symmetrical shape of this figure supplies us with the crystal system of the corresponding crystal body. Every (so-called rationally) constructed extension of the basic harmonic configuration arising through the union of points and intersecting lines leads to lines of

intersection (or points of intersection) of possible crystal faces (or edges of crystals). With regard to the hexagonal crystal system this fact is illustrated in *figure 12*.

These preliminary considerations now enable us to precisely formulate the crystal morphological hypothesis. Its hypothetical character lies in the assumption of *continuity* (in contrast to discreteness), and in the *finite boundary* of crystals by means of *planes*.

Crystal Morphological Hypothesis: *Crystals may be understood as continuous forms, in particular as convex polyhedra bounded by a finite number of planes (faces); the planes and edges of these polyhedra intersect in the ‘plane at infinity’ in the lines or points respectively of a basic harmonic configuration with the rational addition of a small number of steps.*

It must be remembered that when considering the relationship between the hypotheses under investigation here initially only the *geometrical* content is of significance. A morpho-genetic interpretation that complements the traditional structural-genetic interpretation will be examined below. We shall expressly leave open here whether or not this also has a bearing on forces that up to now have not been considered in physics or not thought to strictly belong in the domain of physics.

Why Projective Geometry?

By three-dimensional ‘projective geometry’ we generally understand concepts and relations, whose points, lines and planes or positional relations respectively don’t involve any reference to parallel notions or to metrical concepts and relationships. We will treat this mathematical domain as an independent system of concepts and relations capable of being defined by its own axiomatics. (cf. *Ziegler* (1998), Chapter 2).

Box 6 illustrates in two projectively equivalent representations a figure that is both characteristic and fundamental to project geometry. This figure lies as a configuration in the ‘plane at infinity’ underlying figures 11, 12 & 13b.

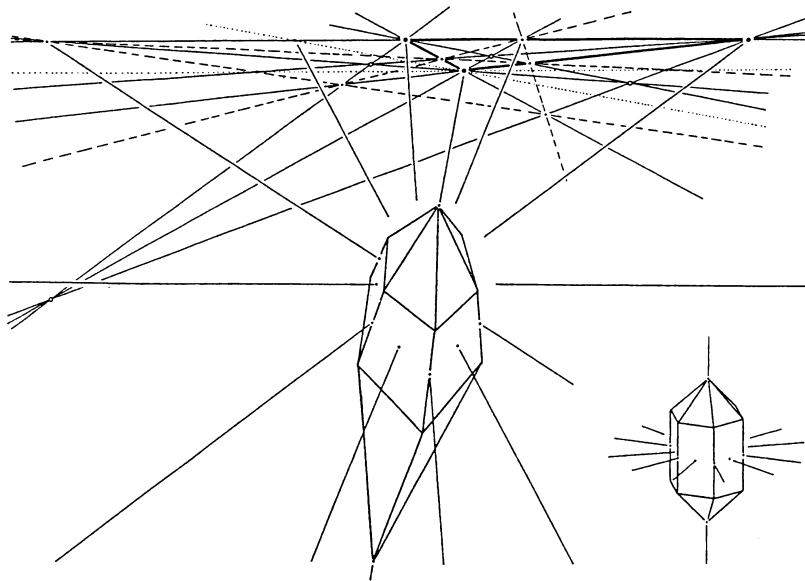


Figure 12: Projective representation of a hexagonal crystal with its axes of symmetry and the correspondingly extended basic harmonic configuration in the ‘plane at infinity’ (from *Ziegler* (1998), figure 6.10, p. 178).

In terms of understanding the geometric crystallography being developed here, three-dimensional projective space serves as the ‘background’, as the structure lying at the basis of all the individual relations. In this sense the *expression ‘projective geometry’ is also used for the embedding of Euclidean or affine geometry in the context of projective geometry*. This characterizes the very quality of the extension used here, in contrast to those viewpoints that are mostly orientated toward Euclidean geometry.

The inclusion of projective geometry of three-dimensional space in crystal morphological investigations may be justified by a number of different reasons:

(a) *Mathematical reason*: As mentioned above, the law of rational indices may be interpreted using the concepts of projective geometry and therefore embedded in a comprehensive mathematical relationship. Thus, in addition to symmetry theory (symmetry groups), projective geometry becomes a significant mathematical foundation for the morphological standpoint, and consequently for geometric crystal morphology. Moreover, because of the symmetry framework, we find that symmetry theory may also be embedded in a projective context and therefore be grasped in its character spanning the whole of space. This may be summarized in the following way: *projective geometry plays a role in the mathematical background of morphology and the symmetry of crystal forms, similar to that of lattice geometry in the mathematical background of structure theory and the theory of space groups*.

(b) *Historical reason*: Projective geometry had a major influence on 19th century crystal geometry, not only enormously contributing to the visual-geometrical attraction of this field, but also to the concrete application of projective methods in the representing of crystals.

At the beginning of the 20th century the link between geometric crystallography and projective geometry that had been developed in the 19th century virtually disappeared from the relevant textbooks without a trace. There are two main reasons for this: firstly, the dwindling interest in the study of projective geometry during the first half of the 20th century; and secondly, the superior place accorded to the lattice structure hypothesis due to the success of x-ray structural analysis and therefore lattice geometry.

(c) *Physical reason*: The assumption of an *infinite* expansion of the crystal lattice finds its natural context in projective geometry. This context at once allows a concrete (not additive or local) connection between the morphological laws and the structural laws.

(d) *Natural-philosophical reason*: A complete morphology of crystal forms for all crystal systems, that is, an insight into their geometric formative principles is also possible without presupposing the lattice structure hypothesis. Here the ideas of projective geometry play a vital role. Therefore we should also strive to *conceive* the lawfulness of the micro-structure as an expression, as a possible consequence of the macro-structure, i.e. as a result of the phenomenological shape of the crystal body, just as one normally pictures the macroscopic shape from out of the microscopic structure. On the basis of reflections of this kind it therefore makes sense to unite these two standpoints into a synthesis (see further below).

For the pragmatic standpoint based on the traditional physical knowledge of Nature, the addition of projective geometry is not seen as relevant. It doesn’t question this manner of perceiving in principle because it doesn’t contradict any of the physical facts. However, the perspective of projective geometry opens up the possibility of also considering crystallography from a non-reductionistic point of view (see the ‘remarks on method’ at the start of the next section).

6. Towards a Complementary Relationship between Morphology and Structure Theory

In this section the morphological principles of crystal forms will be united with the geometrical principles of crystal structure theory. To begin with this will take place on a geometric basis. The aim is to show that the principles of crystal structure and those of general crystal morphology stand in a *complementary* relationship. This essentially means, that starting from the concrete contents of these corresponding principles the one domain cannot be ‘reduced to’ or ‘derived from’ the other, and that both must therefore be united in a synthesis for an objective science of crystals.

Remarks on Method

The following exposition has at base a point of view requiring a few further comments. In order to avoid misunderstandings it must be borne in mind that it is not in any way a matter of calling into question the conventional physically orientated view of crystallography, or of even trying to show that this view is contradictory (i.e. according to its own rules and methods). I do not consider this standpoint (particularly the atomistic view of crystal structure) to be false – but merely one-sided and hence in need of an extension.

To properly demonstrate this point another attitude to knowledge is necessary, an attitude broader and deeper than the one prevailing today, especially among the natural sciences. In my opinion the gulf between the contemporary views of science and an anthroposophically extended natural and spiritual science cannot be bridged (if such a bridge is striven for at all), as long as we remain fixed in that pragmatic attitude to knowledge held by natural scientists and which is primarily orientated towards measurable results. In the first instance this attitude is concerned with the construction of idealized models (aided more or less by complex mathematical methods), which are able to represent or simulate all the empirical findings. A model is designated as appropriate as long as it is compatible with other (accepted) models and doesn’t contradict any of the empirical data. Furthermore, all the consequences capable of being derived from this model must harmonize with the physical facts. Whether the mere indirectly accessible or inferential particularities of this model, its inner presuppositions, its inner structures etc. directly relate to the experiential reality or not, is only secondary for this viewpoint, and strictly speaking even irrelevant. As long as the model adequately simulates the given data and we are capable of making precise and verifiable predictions, there is no reason to call it into question – a fact entirely independent of the model’s composition and structure.

Nevertheless, for the cognitive point of view advocated here concepts and ideas (conceptual relations) are not simply a medium for constructing a model, but important principles for gaining insights into the constitution of experiential reality. And this can shed new light upon the idealized assumptions (ideas) of mathematical-physical models, since they may now be understood as necessary and objectively introduced ideas, and not simply as arbitrary settings serving to adapt the model to the empirical data or mathematical conditions.

One of these assumptions – the theoretical infinite nature of a lattice – will be more closely examined in the following section.

Thus we will not attempt to refute the standpoint of physical crystallography in any of its details, but simply try to bridge the gulf mentioned above. This will be done by showing how the physical point of view may be supplemented through a view potentially embracing within it experiences other than merely physical and sensible ones.

Law of Crystal Forms and the Crystal Morphological Hypothesis

The law of crystal forms cannot be derived from the crystal morphological hypothesis, which, according to its geometrical content is equivalent to the zonal relationship law. For this hypothesis does not address the symmetry relations between crystal polyhedra, or to be more exact between corresponding tangential polyhedra. The symmetry relations for crystal polyhedra are provided by the crystal symmetry laws (*box 4*).

Those symmetry groups solely compatible with the crystal morphological hypothesis (or the zonal relationship law respectively) are precisely the 32 crystallographic point groups underlying the above mentioned simple crystallographic facial forms. Hence it is clear that the law of crystal forms may be derived from the crystal morphological hypothesis (or the zonal relationship law respectively) together with the crystal symmetry laws.

On the Relationship between the Law of Crystal Forms, the Crystal Morphological Hypothesis and the Lattice Structure Hypothesis.

The *lattice structure hypothesis* and the *crystal morphological hypothesis* cannot be derived from one another. For neither the law of crystal faces nor the fact that crystals behave like a continuum may be inferred from the lattice structure hypothesis. On the other hand, using the assumptions concerning the relation between the size of crystal particles and the polyhedral shape, the lattice structure hypothesis is thoroughly *compatible* with the assumption of planar faces (law of crystal faces) in the sense of the law of correspondence. In contrast to the crystal morphological hypothesis, however, the law of crystal faces is *not* contained in the lattice structure hypothesis – it only results from the combination with the preferred (in terms of energy) densely packed lattice planes.

Furthermore, we also cannot derive from the crystal morphological hypothesis the fact that crystal bodies consist of lattice-shaped, i.e. threefold translational-periodically arranged particles of finite size and finite distance. And again using the assumptions about the relation between the size of the particles and the phenomenological shape – the crystal morphological hypothesis is compatible with a lattice-shaped construction of this kind.

In addition, it must be kept in mind that neither of these hypotheses is true without some sort of restriction: every experiment furnishing an indirect confirmation of the lattice structure hypothesis also demonstrates that this hypothesis is not perfectly fulfilled (lattice faults and defects, inclusions, mosaic lattices: reciprocally dislocated lattice blocks). On the other hand, an examination of every real crystal shows that the planar nature and straightness of the faces and edges is also only approximately fulfilled. Here too neither of the two hypotheses dominates.

The development of the *law of crystal forms* from the lattice structure hypothesis on the basis of both the *law of crystal faces* and the *law of correspondence*, that is the assumption that *planar* boundary faces tend to appear in densely packed lattice planes, belongs to the standard fare of modern crystallography and symmetry theory. From the mathematical standpoint it is crucial to understand the 230 space groups as symmetry groups of all types of *infinitely* expanded lattice complexes, and the 32 crystallographic point groups as those subgroups of space groups that are introduced to leave such a lattice and *one* of its points invariant. This means that the crystal symmetry laws (*box 4*) *can only* be derived from the lattice structure hypothesis if we proceed from the hypothesis of an *infinite* crystal lattice filling the *whole* of space.

If we solely confine ourselves to the conditions of an infinitely expanded lattice, then strictly speaking there is no longer such a thing as morphology. This principal objection isn't invalidated by reference to the 'extremely large' number of atoms in a crystal body, which from the structural standpoint appears to be 'practically infinitely large'. Without any sort of

pretension to knowledge the latter may be a useful working hypothesis for obtaining numerical results and forming models. Yet in the sense of the above ‘remarks on method’, if we demand more from a model than a simple harmonization between its theoretical-computational consequences and the empirical facts, then we must also demand that all its defining properties agree with the observation. For instance, it is therefore rather typical of crystals that they occupy a well-bounded finite area of space. Moreover, considered mathematically, there also exists a principal difference between the finite and the infinite, and not simply a gradual or approximate difference, however large ‘finite’ may actually be.

As a result of the foregoing considerations, if we then proceed from a *finite* lattice it is only possible to derive the *law of crystal forms* from the *lattice structure hypothesis* in conjunction with the *law of crystal faces*, the *law of correspondence* and the *crystal symmetry laws*.

The geometrical nucleus of the crystal morphological hypothesis is equivalent to the zonal relationship law or the fundamental law of crystallography. Thus the *law of crystal forms* follows from the *crystal morphological hypothesis* only in conjunction with the *symmetry laws*, i.e. together with the fact that the tangential polyhedra defined by this hypothesis are *symmetrical*.

Form Synthesis for Finite Lattices

If we join the crystal morphological hypothesis together with the symmetry law and lattice structure hypothesis in such a manner that only those point formations are allowed to be lattices that are possible according to the basic harmonic configuration set out in the crystal morphological hypothesis (the so-called Möbius lattice, see *box 8*), then for a finite lattice we obtain the following:

Form Synthesis (Finite Lattice): *The law of correspondence follows from the lattice structure hypothesis, the crystal symmetry laws and crystal morphological hypothesis.*

For on the one hand the law of crystal forms may be derived from the crystal morphological hypothesis with the help of the symmetry law, while on the other, by virtue of the lattice structure hypothesis, the crystal morphological hypothesis restricts the possible macroscopic forms precisely to those polyhedra whose faces are densely packed lattice planes or lattice planes of a space lattice. The latter follows from the fact that because the rational extension of the basic harmonic configuration in the ‘plane at infinity’ is restricted to a *small number* of steps, precisely those lattice planes are chosen which are most densely packed.

Form Synthesis for Infinite Lattices

We are now faced with the following situation: *either* we hold fast to the finite nature of the crystal’s lattice structure, thereby necessarily renouncing not only a mathematical theory of structure but also the founding of a morphology based on such a theory. *Or* we postulate the infinite nature of a crystal lattice as the basis of crystal physics with the avowal that we are not in a position to directly establish it empirically in the usual sense.

At heart it is not a question of striking a preference for one of the two alternatives: the assumption of an *infinitely* extended lattice is *fundamental* to the entire theoretical physics of crystals, and an especially important basis for the relevant theory of space groups in crystal physics.

Nevertheless, the assumption of a crystal lattice spanning the whole of space cannot be justified on the basis of traditional *physical* fundamental principles alone, for it is neither present as an empirical fact, nor can be conceived in its totality as a result of the assumed reciprocally active forces (the latter only establish that *locally*, lattice-shaped arrangements of

particles are preferred in terms of energy). If such particles are absent, then the corresponding lattice structure also cannot be established.

Consequently, this assumption must be justified in another way, even if it is perhaps already viewed as established in a pragmatic sense because the corresponding model supplies useful predictions.

Interestingly enough, it is precisely the crystal morphological hypothesis that offers a suitable geometrical context for a lattice of this nature – i.e. one that is structured throughout the whole of space – and not merely for that portion of it that is ‘occupied’ by tiny particles (see section 7.2 in Ziegler (1998)). This means that the assumption of the lattice structure hypothesis really only makes sense *together* with the assumption of the crystal morphological hypothesis. Thus, the assumption of an infinite lattice yields the crystallographic symmetry laws: the 32 crystal classes are precisely those symmetry groups leaving a lattice and one of the lattice’s points invariant.

With regard to the lattice structure hypothesis, if we proceed from an *infinitely* extended lattice underlying the crystal, then we can derive the symmetry laws and hence the law of crystal forms as well. We do this by integrating the crystal lattice in the determined spatial configuration, using the morphological hypothesis with its basic harmonic configuration in the ‘plane at infinity’ (Möbius net, see *box 8*). As indicated in the foregoing discussion on the finite lattice, by virtue of the lattice structure hypothesis, the crystal morphological hypothesis restricts the possible macroscopic forms to precisely those polyhedra whose faces are lattice planes of the space lattice and densely packed with points (cf. Ziegler (1998), sections 7.2 & 7.3). These presuppositions also yield the law of correspondence.

Box 8: Möbius Nets and Möbius Lattices

The construction of a complete quadrilateral or quadrangle (*box 6*) can be continued in such a way that ultimately (after infinitely many steps) a simple pattern arises covering the entire plane (*figure 13a*), a so-called *Möbius net*. It corresponds to the Euclidean parqueting of the plane by parallelograms, in particular rectangles, quadrates and rhombi. If we start with a complete quadrangle $ABCD$ with 6 sides and an adjoining triangle RST , then a side s of the triangle RST with its 4 points of intersection $MRNT$ with the 6 sides remains fixed.

Essentially, a basic harmonic configuration in the ‘plane at infinity’ is determined by means of a cubic-shaped starting cell (*figure 11*). It then follows that an unbounded connected spatial array of cells of this kind – a so-called *Möbius lattice* (*figure 13b*) – may be uniquely constructed by extending and continuing the construction of the corresponding Möbius net. This may be understood as a projective multiplying of the starting cell.

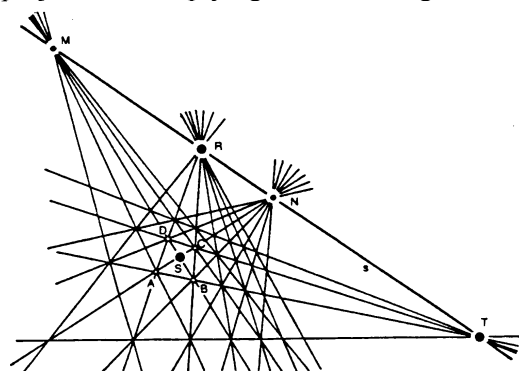


Figure 13a: Möbius net
(from Ziegler (1998), figure 7.1, p. 208)

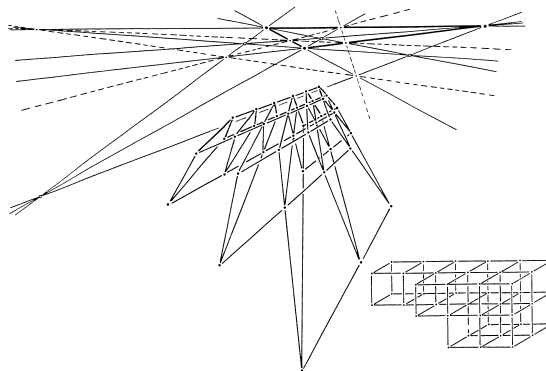


Figure 13b: Möbius lattice
(from Ziegler (1998), figure 7.2, p.209)

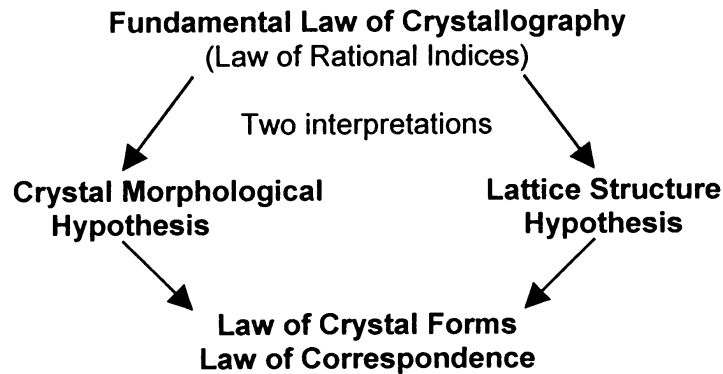
Form Synthesis (Infinite Lattice Structure): Both the law of crystal forms and the law of correspondence follow from the synthesis of the lattice structure hypothesis and the crystal morphological hypothesis.

Complementarity of Morphology and Structure Theory

It is interesting, and not only significant from the mathematical point of view, that when considered in isolation, neither the lattice structure hypothesis (both for finite and infinite lattices) nor the crystal morphological hypothesis, is sufficient to determine the law of crystal forms. Moreover, both these hypotheses are not mutually exclusive, but supplement each other in a *complementary* fashion, in the sense that one of them contains conceptual components that together with the other (and in the finite case in connection with both the crystal symmetry laws and the law of correspondence) constitutes the law of crystal forms.

Thus an objective science of crystallography hoping to give all the known phenomena their due, can neither dispense with the one nor the other hypothesis (box 9).

Box 9: Complementarity of Morphology and Structure Theory



In summary the following holds: *The crystal morphological hypothesis and the lattice structure hypothesis stand in a **complementary** relationship.*

A Formative Principle for Crystals

If we start from the fact that the principles constituting a crystal polyhedron permeate the entire crystal, then it is obvious to say that the formative principles appearing in its external shape (planar faces, symmetry of tangential polyhedra, ‘distortion’) also underlie its inner formation.

If we understand lattice points as bearers of centrally harmonic bundles of planes, each containing the basic harmonic configuration (and their rational supplements) in the ‘plane at infinity’ underlying the lattice, then instead of the idea of a point lattice consisting of locally connected particles, we have the idea of a lattice consisting of formative centres that are distantly (globally) connected by planes.

Together with the crystal symmetry laws we therefore obtain the formative principle for crystals.

Formative Principle for Crystals: Crystals may be understood as continuous forms, in particular as convex polyhedra bounded by a finite number of planes (faces) and whose tangential polyhedra are symmetric. These faces consist of planes of the basic harmonic configurations (with a few rational additions) which are concentric and immanent in the crystal body and whose centres lie in the points of a lattice, and whose planes contain, in the ‘plane at infinity’, the basic harmonic configuration underlying the lattice.

The conceptual components of this principle are sufficient to derive the law of crystal forms.

7. Physical Points of View

In this section we will discuss some physical points of view that may lend support to the previous considerations. Naturally, it is not possible to more closely examine all the conceptions that will be mentioned.

The lattice structure hypothesis for crystal bodies discussed in the previous section turns out to be particularly important in interpreting the phenomena of growth, the constituents of the outer-surface, faults, dislocations, the spiral structure etc. In investigations of this kind, the information contained in the hypothesis regarding the precise arrangement of the particles in space (especially the content of the unit cell) proves to be significant.

Yet this is not the case for many of the properties treated in the solid-state physics of ideal crystals concerning dynamic, thermal and electrical processes. When devising physical theories for properties of this nature, the location in space with its information about the spatial arrangement of the particles is scarcely a factor. Here so-called ‘Fourier space’ plays a highly important role (see for example *Kittel’s* (1999) standard textbook).

Symmetries of Laue Diffraction Images

From a crystal polyhedron’s given configuration of points and lines in a plane of projection, particularly in the ‘plane at infinity’, i.e. from both its symmetry framework and its faces and edges, it is impossible to know whether or not the symmetry group of the corresponding tangential polyhedron contains the inversion (point reflection). This is because the plane configuration is invariant with respect to the inversion. Hence on the basis of the configuration alone we can only tell if the symmetry belongs to one of the 11 crystallographic groups with a centre of inversion, or to one of the subgroups of these groups.

In specific experimental arrangements (Laue diagrams), the diffraction images of crystals using electronic, neutron or x-rays, generally exhibit the same behaviour. The symmetries of the point distributions in these crystal diffraction images are exactly the same as the symmetries of the above mentioned crystal polyhedral configuration in the ‘plane at infinity’. In connection with the x-ray structural analysis of crystals, the 11 crystallographic groups are called *Laue classes* or *Friedel classes*.

Under the physical conditions of x-ray diffraction (or the diffraction of other ‘rays’) crystals manifest the primary symmetry components of their formative principles. Hence with respect to their symmetric quality, diffraction images may be interpreted as the representation of the symmetry relations in the ‘plane at infinity’, penetrating into the inside of the crystal.

Geometry of Lattice Planes

An investigation of the conceptual interpretation of these experimental findings in the context of conventional solid-state physics, particularly in the domain of crystal diffraction, shows that structures determined by *planes* are actually of great importance. As we shall see, it is also quite natural to embed these structures into projective space.

In the interpretation of Laue diffraction images *Ewald* (1921) introduces the so-called *reciprocal lattice* of a primary lattice (also called a *direct lattice*). This is a representation of the space of lattice planes by the points (hkl), with the indices hkl from the direct lattice (cf. *Buerger* (1977), sections 4 & 5; *Kleber* (1990), section 5.1.4; *Vainshtein* (1994), section 3.4.3).

The lattice planes of a lattice form sets of parallel planes, each having a constant distance and weight (point or packing density) within the set. Every one of these sets corresponds to *one* point of the reciprocal lattice; this point's location vector is perpendicular to the planes of 'its' set of lattice planes; its distance from the origin is inverse to the value of the distance for the planes of the corresponding set of lattice planes.

One and the same lattice can accordingly be described by the positions of its points or by the positions of its lattice planes. Every set of lattice planes contains all the points of the lattice, hence representations from all the sets of lattice planes of a lattice pass through any arbitrary point of the lattice. They form a bundle of planes with the same point-symmetry as the original direct lattice. Furthermore, the lattice planes intersect the 'plane at infinity' in configurations that arise from rational constructions starting with the basic harmonic configuration, which contains the intersecting elements of the symmetry framework (cf. Ziegler (1998), chapter 7.2). Consequently, the geometry of lattice planes is determined by the basic harmonic configuration that is characteristic for each of these crystal systems (apart from the purely metrical properties).

The following holds in the transition from a lattice to a lattice-complex (a composite lattice): (1) The arrangement of the sets of lattice planes do not change, i.e. every lattice plane of a partial lattice is also a lattice plane of the lattice complex, and every lattice plane of a lattice complex is also a lattice plane of one of its partial lattices. (2) The weight and distances between the planes of a parallel set of lattice planes are not equal, but *periodically* arranged. Thus the corresponding reciprocal configuration is no longer a lattice having a translational periodicity, but a *weighted* lattice – the so-called *reciprocal structure*. (3) The representations of the sets of lattice planes through any arbitrary point of a lattice complex form a bundle of planes with the same point-symmetry as those belonging to the crystal system of the direct lattice.

Bundle Structure Hypothesis

The observational results, particularly from crystal diffraction experiments, do not therefore really correspond to the spatial information about a lattice, but to the character and distribution of the sets of lattice planes – represented by the reciprocal structure. This interpretation is based on the following hypothesis:

Bundle Structure Hypothesis: *Crystals may be understood as sets of parallel planes possessing a periodic distribution of 'density' and distance.*

Both the reciprocal structure representing this bundle structure and the corresponding reciprocal lattice play a central role in the Fourier representation of lattice-periodic functions (charge density, electron density etc.). Essentially, the bundle structure hypothesis is geometrically equivalent to the lattice structure hypothesis and may be easily integrated with the principle of crystal formation formulated in the previous section. However, its pre-eminent place in the solid-state physics of ideal crystals is due to the fact, that any interpretation of a crystal as a composition of elementary particles in a periodic array, has to give way to an interpretation of the crystal that emphasizes the combined working of sets of parallel planes.

Therefore this once again justifies the use of projective geometry as an all-embracing background: for only in projective geometry can planes – in addition to points – be considered as independent and fundamental elements. This means that planar formations, and consequently projective geometry, have a role to play in both the conceptual understanding of morphology, as well as in structure theory.

8. Crystal Genesis

By crystal genesis we mean the origin of crystal polyhedra. Up to now we have mainly argued geometrically and concentrated on the final, developed form, i.e. its shape and lawfulness, without a consideration of the genetic factors and formative forces. Let us now outline a few basic reflections on the genesis of crystals.

Every investigation of crystal genesis must take into account the basic principles of its form and structure. For this reason, crystal genesis will be treated as the problem of *genetically interpreting* those two hypotheses discussed in the previous sections – the crystal morphological hypothesis and the lattice hypothesis. Moreover, the complementary character of these hypotheses suggests that we can also establish the corresponding complementary genetic principles (for a more detailed treatment of this point see Ziegler (1998), section 7.7).

Structural Genesis of Crystals

In the study of the structural genesis, we begin with an interpretation of the fundamental law of crystallography using the lattice structure hypothesis. On the basis of this interpretation, and from the standpoint of traditional physics, all the macroscopic properties of crystals are considered to be a consequence of the micro-structure, or to be more precise: are understood as being compatible with the lattice structure hypothesis.

To summarize, we arrive at the following genetic interpretation of the lattice structure hypothesis in conjunction with the law of crystal faces and the law of correspondence:

Structural-Genetic Principle of Crystal Development: *According to the combination of a finite number of particles, crystals develop in a local and successive (additive) manner into lattice complexes with an external polyhedral shape, on the basis of the properties and laws of reciprocal effects of these particles, as well as on the basis of the principle of energy minimization.*

Morphogenesis of Crystals

When studying the morphogenesis of crystals, we start with an the interpretation of the fundamental law of crystallography using the crystal morphological hypothesis. On the basis of this interpretation, all the (sub)-microscopic properties of crystals are considered to be the consequence of morphological principles, or to be more precise: are understood as being compatible with the crystal morphological hypothesis.

To summarize, we arrive at the following genetic interpretation of the crystal morphological hypothesis in conjunction with the law of correspondence:

Morpho-Genetic Principle of Crystal Development: *Crystal polyhedra develop in an entirely uniform manner, according to the configuration of the infinitely distant lines of the faces under the conditions of being a part of a basic harmonic configuration, and in their inner structure according to the corresponding lattice.*

From the standpoint of the *morphogenesis of crystals*, which is complementary with the conventional one, it is important that the essential elements of the lattice structure (connected with a crystal polyhedron) prove to be the expression of the morphological features, i.e. that they may be *conceived* as a possible consequence of the form, and not necessarily as the ‘cause’ of the form.

Genetic Synthesis

If we also assume the crystal symmetry laws, then on the basis of a genetic interpretation of the principle of crystal forms, we arrive at the following synthetic principle of crystal development:

Crystal-Genetic Principle: *Crystals are formed by finitely many planes (faces) into continuous convex polyhedra, whose tangential polyhedra are symmetrical. These faces lie in planes of the basic harmonic configurations (with a few rational supplements) immanent in the crystal, whose centres lie in the points of the lattice, and whose planes contain the basic harmonic configuration in the 'plane at infinity' underlying the lattice.*

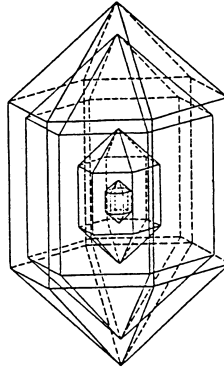


Figure 14: Quartz crystal showing its step-wise growth
(from Borchardt-Ott (1997), figure 5.5, p. 34)

9. Conclusion and Outlook

The present investigation shows that assuming an *infinite* expansion of the crystal lattice for the interpretation of different kinds of experimental results, at once furnishes a justification for broadening the viewpoint of crystallography: this assumption demands the development of a morphology on a projective basis that is complementary with structure theory, as we have tentatively outlined here.

In conclusion we shall refer to a few further aspects of crystallography, which cannot be expanded upon at the present time. In particular, there does not currently exist any specialized morphology of natural crystal forms based on the viewpoints set out in this paper.

A central area of specialized morphology is the morphology and metamorphosis of forms of growth. For a given mineral, or any general crystallized substance, the actual final shape of the crystal polyhedron (form and habit) as well as the intermediary forms that emerge in the course of these processes, considerably depend on the conditions in which they arise (concentration of the crystallizing substance, presence of other substances in the solution and their rate of concentration, relations for the temperature and pressure etc.), and not so much on the physical and chemical properties of the substance. The latter especially come to expression in the concrete shape of the lattice in the form of a lattice complex (composite lattice) with its unit cell, that is, in the specialized way in which the substance is arranged within the general symmetry relations (represented by the crystal system and crystal class). Both the morphology of growth forms of particular substances, as well as their actual lattice structure, underlie the respective crystal system or crystal class as a higher order, as a comprehensive formative principle.

By extending the view that commonly concentrates on the lattice structure, the independent and complementary formative principles underlying the morphology allow the world of crystals to be embedded in an all-embracing relationship. This comprehensive view which encompasses not only the morphological but also the structural formative principles,

does greater justice to our immediate experience of the beauty, harmony and purity of crystals, than a confinement to the lattice structure.

There naturally emerges here a whole host of further tasks and problems whose resolution is still in its infancy. For example, how do the morphological formative principles relate to the corresponding formative principles in the other kingdoms of Nature, or even to the domains of lifeless Nature? Are the known physical effects of force connected with the morphological principles, or do other forces play a role? These questions must remain unanswered here.

References

- Adams, George* (1934): Die Kristallgestaltung des Raumes. In: Strahlende Weltgestaltung. Synthetische Geometrie in geisteswissenschaftlicher Beleuchtung. Dornach, Mathematisch-Astronomische Sektion am Goetheanum. (2nd ed. Dornach 1965), chapter 7, pp. 325-386.
- Adams, George* (1957): Das 'reziproke Gitter' und die Röntgenanalyse der Kristalle. Mathematisch-Physikalische Korrespondenz, No. 12, p. 48. Reprinted in: *G. Adams* (1996): Universalkräfte in der Mechanik. Dornach; Mathematisch-Astronomische Blätter (Neue Folge, volume 20, pp. 213-224).
- Borchardt-Ott, Walter* (1997): Kristallographie. Eine Einführung für Naturwissenschaftler. Berlin (5th ed.). [English translation of 4th edition: Crystallography. Springer Verlag, 1993, Berlin.]
- Buerger, Martin* (1977): Kristallographie. Eine Einführung in die geometrische und röntgenographische Kristallkunde. Berlin.
- Ewald, P. P.* (1921): Das 'reziproke Gitter' in der Strukturtheorie. Zeitschrift für Kristallographie, Volume 56, pp. 126-156.
- Kittel, Charles* (1999): Einführung in die Festkörperphysik. Munich. (12th ed.).
- Kleber, Will* (1990): Einführung in die Kristallographie. Berlin (17th ed.).
- Landau, L. D. & Lifschitz, E. M.* (1979): Statistische Physik, Teil 1 (Lehrbuch der theoretischen Physik, vol. V). Berlin (5th ed.).
- Niggli, Paul* (1949): Probleme der Naturwissenschaften, erläutert am Begriff der Mineralart. Basel.
- Parker, Robert* (1975): Mineralkunde. Thun (5th ed.).
- Shubnikov, Alexej. V. & Koptsik, Vladimir A.* (1974): Symmetry in Science and Art. New York.
- Steiner, Rudolf* (1918): The Philosophy of Freedom. Rudolf Steiner Press, London. Translated by Michael Wilson.
- Vainshtein, Boris K.* (1994): Fundamentals of Crystals: Symmetry, and Methods of Structural Crystallography (Modern Crystallography, Volume 1). Berlin/Heidelberg (2nd ed.).
- Ziegler, Renatus* (1998): Morphologie von Kristallformen und symmetrischen Polyedern. Kristall- und Polyedergeometrie im Lichte von Symmetrielehre und projektiver Geometrie. Dornach. Mathematisch-Astronomische Blätter, Neue Folge, vol. 21.

APPENDIX

A. Rudolf Steiner on Crystals

The following tables contain all the references known to me in the work of Rudolf Steiner in which he writes or speaks about crystals in some way. The dates enclosed in square brackets do not mention crystals, yet treat of topics that I believe are important for a deeper understanding of crystals. Since crystals are the archetypal form of the mineral kingdom (22. 9. 1918), it is possible to enlarge this collection to include all the places bearing any sort of reference to mineralization and minerals etc. However, because Steiner frequently spoke or wrote on this theme it would entail referring to a huge number of passages in the written works and therefore be virtually impossible to survey.

The dates of especially detailed or significant places are indicated in bold face. Steiner's expositions in *Theosophy* (1908) and the lectures from 7.4.1921 & 13.8.1924 belong among the most central references.

The following section contains a few passages that discuss crystals and crystallization. Only the dates are given as all further information may be obtained from the tables. The polyhedral shape having its origins in lunar forces is especially important for crystals (5.11.1921), in contrast to the rounded, spherical forms of the organic kingdom (7.6.1905, 7.4.1921, 5.11.1921). Crystals essentially constitute a whole: crystallography is an empirical science having its basis in reality (15.8.1916, 29.12.1919). Yet a more exact observation discovers that the entire cosmos, the entire universe with all its beings is also involved in the development of crystals (25.10.1906, 9.8.1922, 21.6.1924, 13.8.1924).

In the mineral kingdom crystallization does not form an exception but is the rule, at least in accordance with its tendency (22.9.1918, 22.8.1910, 13.1.1923). The earth is most inclined to crystallization in the so-called 'warmth-night' (7.3.1920). Finally, the mineral world has come into being out of the solidification or ossification of the plant world (8.12.1910, 9.2.1911).

Steiner has set down the main aspects of crystal development in his book *Theosophy* (1908):

"And crystal forms stand between the shapeless type of mineral matter that we find in gases and fluids, and the living shapes of the plant world. We find in crystals the transition from the shapeless mineral world to the formative ability of the plant kingdom. – In this formative process present to the external senses, we behold in both the mineral and plant kingdoms the sensible solidification of a purely spiritual process, that occurs when the spiritual seeds of the three higher regions of spirit land develop into the spirit shapes of the lower regions. In the spiritual world, the process of crystallization corresponds in its archetypal image to the transition from the formless spiritual seed to the *formed* structure. If this transition solidifies in such a way that the senses can perceive its outcome, then this is exhibited in the sense world as the process of mineral crystallization." (*Theosophy*, 1908, pp. 149-151).

"The *first region* [of spirit land] contains the archetypes of the physical world, insofar as it is devoid of life. The archetypes of minerals are also to be found there [...]. The objects and beings embodied in the physical world are present there as 'hollow spaces', while the mobile activity of the archetypes (and the 'spiritual-music') takes place in the intervening spaces. In the process of physical embodiment the hollow spaces become more or less filled. Whoever is able to perceive this space with both physical and spiritual eyes would see physical bodies, and between them the mobile activity of the creative archetypes." (*Theosophy*, 1908, pp. 124-125).

Rudolf Steiner refers to this account in many different lectures, for example, on the 6.8.1908, 22.8.1910, 18.1.1912, 8.3.1920, 13.8.1924. The centre of crystal forming spiritual forces may be found on the Arupa-plane (higher Devachan) (19.10.1905, 4.12.1907). These forces originate from regions lying far beyond our solar system (12.8.1908, 14.4.1912). World-Thoughts are embodied in crystals (13.1.1923).

Steiner likewise discusses the cosmic configuration of forces underlying the formation of crystals on many different occasions (cf. for example, 6.8.1908 for a reference to the etheric body, astral body and ego of a mineral). Also interesting in this connection is the statement that matter is condensed light (27.5.1910). The following couple of passages are cited because they appear to be particularly helpful in understanding the themes under discussion here.

“*Crystals* result from complicated encounters of force; atoms from simpler ones. – *Amorphous masses* arise from the neutralization of the directions of force. – *Force* is the manifestation of spirit viewed in a one-sided spatial manner. We cannot say that force works upon matter, since matter only consists of the array of effects from the converging rays of force. [...] – *Mathematics* is the abstracted sum of forces that are active in space.” (1919).

“Certain physicists nowadays have as much right to say that they have photographed the configuration of atoms, as spiritists do when they claim to have photographed spirits. Certainly one can bombard crystals with x-rays; one can reflect the x-rays, bring the reflected rays into interference and photograph this, and claim to have photographed the configuration of atoms. However, the crucial point is this: does one really photograph atomistic agents or merely certain effects stemming from the macrocosm, that only show up as points, leading one to believe that atoms are present? The fact is we must discover methods of conceiving and thinking that are capable of proceeding in the right way from the phenomena to the actual being of things.” (24.3.1920).

“In reality, our spiritual scientific studies lead us to view a crystal in such a way that we do not merely derive it from certain inward plastic forces present in some sort of material substance, but that we relate it to the outer aspect of the cosmos, and seek the directions in the cosmos which yield the individual crystal form through the distribution of their points of origin. In the individual crystal form we really receive, as it were, impresses of mighty cosmic relationships. This will be studied in all its details.” (7.4.1921)

“If we bring the archetypal shape of the mineral kingdom before our eyes, then it is the crystal shape, the polyhedral shape (see drawing [Plate 13: left octahedron, right oblong quadrate]); the mineral somehow confronts us as polyhedral in nature. This polyhedral shape confronts us in one kind of mineral in one particular form, in another mineral in a different form; it cannot be comprehended in any other way than by initially beholding the mineral, that has formed itself out of the forces active within the mineral realm. We have to imagine: if we have any sort of mineral extended in length, then the forces working in this direction (right in drawing), are suitable for extending the mineral lengthways. While the forces working in this direction (right, horizontal line in the middle), perhaps unfold diminished in strength – or as we would then express it – that make the mineral narrower in this direction and so on. In short, so as to be able to speak about minerals at all, whether the forces are active from without inwards or within outwards, we have to imagine that these forces stand at angles to one another, that they work in certain directions. We have to above all imagine that these forces are really present in the universe, and active within the earthly domain.” (5.11.1921, pp. 140-141)

“In the universe everything is ordered in a crystalline fashion. What we see there (in crystals) is of such a nature because the forces in the universe are ordered just as they are in crystals. One type are active in this way, another in that way, and thus crystals are fashioned out of the entire universe.” (9.8.1922, p. 56)

“As for minerals, we are not only compelled to look at what is above us, but also to the entire periphery [Figure 15]. [...] We glimpse on all sides converging cosmic currents. They encounter one another. [...] However, if we contemplate a current for the mineral kingdom, we also have here a counter-current, and through this interplay there emerges the form of the mineral kingdom. Here we have a current, there we have the counter-current; here again a current, there the counter-current, and so on. And thus the mineral arises as a result of the free interplay of these cosmic currents. The earth is not a mirror for the crystallized mineral. Nothing is reflected back into the earth. Everything is reflected back into its own inherent element.

If you cast your eye upon a mountain and find a quartz crystal, it is usually situated upright on its base; yet this is only because it has been disturbed by earthly factors, Ahrimanic forces have destructively intruded. In reality it would be formed in such a way so as to be

pressured by the spiritual element from all sides, and reflected back upon itself; and you behold the quartz crystal freely floating in spiritual cosmic space. In every single crystal, which is completely formed on all sides, we can behold a tiny world. – Yet there are many kinds of crystal forms: cubes, octahedra, tetrahedra, dodecahedra, rhombic, dodecahedral, monoclinic and triclinic shapes – there exist all manner of possible shapes. We behold them. We see how the currents converge, and encounter one another. Here we have a quartz crystal, a hexagonal prism, enclosed by a hexagonal pyramid; there we have a salt crystal, that is perhaps cubic in shape; here is a pyrite crystal, that is perhaps dodecahedral in shape. We behold all these crystals. Every one of them comes into being in the way I have described, and we must say: just as there are many differently formed world-currents, so there exist many spatial worlds; there isn't just one world, there are as many spatial worlds as the earth is composed of crystals. – We discover an infinite number of worlds. When we see a salt crystal we say to ourselves: out there in the universe it is essential being; for us, the salt crystal is a manifestation of something that permeates the essential being of the universe, that is a world unto itself. [...] In the infinitely diverse forms of crystals, we perceive the manifestation of a multitude of beings, revealing themselves in mathematical-spatial crystal shapes. In crystals, we gaze upon the Gods.” (13.8.1924)

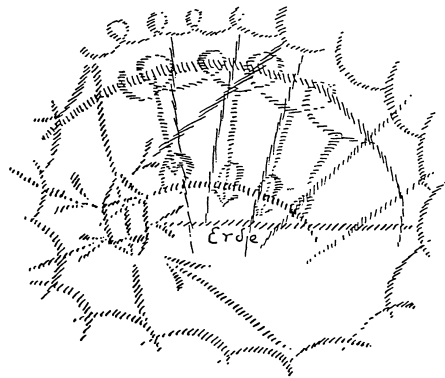


Figure 15: Plate 3 from Rudolf Steiner, “True and False Paths in Spiritual Investigation”, lecture 3, 13th August 1924 (GA 243). (Erde = earth)

The applicability of mathematics to the mineral world is discussed at various times, e.g. on the 18.3.1921, 21.3.1921, 7.4.1921, and 5.11.1921 and on 13.8.1924. On the 21.3.1921 Steiner particularly emphasizes the significance of synthetic projective geometry as a means for penetrating into the inner configuration of spatial forms:

“Synthetic geometry really shows that we can penetrate right into the geometrical forms, something not possible in analytic geometry. [...] Hence we pass from the surroundings of the line, from the surroundings too of the spatial formation, right into their very inner configurations. We are hereby given the possibility of developing the first stage in the transition from pure mathematical knowledge – present in the most eminent sense in analytic geometry – to Imaginative knowledge. Naturally, we do not possess Imaginative knowledge in synthetic projective geometry, yet we approach it [...]” (21.3.1921)

Reference Tables - Rudolf Steiner on Crystals

Date	Year	Place	GA	Edition	Page	Content in Key Words
Theosophy	1904		9		35, 37-38 125-126, 140, 149-153	Life forces and mineral forces Archetypes of the mineral kingdom in the 1 st region of spirit land: they hover between the 'hollow spaces' which are called forth by the physical formations there. Process of crystallization, crystal shapes (4 th Elementary kingdom).
The Stages of Higher Knowledge	1905		12		21	At the stage of Inspiration one perceives the form of crystals as sound.
7. June	1905	Berlin	324a	1995	81-82	Planar-faced boundaries distinguish crystals from living beings.
19. Oct.	1905	Berlin	93a	1987	160-161	The forces that form crystals are to be found on the Arupa plane. One cannot perceive them on the physical plane.
28. Dec.	1905	Berlin	[91]		Last 1/3	Immersion in the world of crystals leads to the Spirits of Wisdom, i.e. over and beyond the Spirits of Movement which underlie the plant world.
25. Oct.	1906	Berlin	55	1983	52-53	The entire cosmos lives in a crystal form, it is an individual impress of the cosmos, not an isolated configuration.
25. May	1907	Munich	99	1979	27	Formation of ice in water
4. Dec.	1907	Munich	98	1996	130-134, 144-146	[Elementary kingdoms] Only the physical body of minerals and crystals is found on the physical plane. The mineral's ego resides on the Arupa plane. The formative forces of minerals and crystals are to be found in higher Devachan, they work from out of the formless realm (1 st Elementary kingdom).
26. Feb.	1908	Berlin	266/ 1	1995	331	Principle for crystallization and dissolution
6. August	1908	Stuttgart	105	1983	54-56	Crystals: etheric hollow-space, etheric light figures; astral light figures in hollow-sphere; rays join together in the ego of the mineral world [periphery], in higher Devachan. Everything solid has at one time crystallized itself out of the fluid earth.
12. August	1908	Stuttgart	105	1983	137	Formative forces of crystals originate from a realm lying far beyond the solar system.
Occult Science – An Outline	1910		13	1989	112, 313	Formative forces of crystals originate from the first region of spirit land; The crystal as meditative content.
[27. May	1910]	Hamburg	120	1992	192	Everything material on earth is condensed light.
[11. June	1910]	Oslo	121	1982	88-92	The solid surface of the earth is a result of counter-currents of force.
22. August	1910	Munich	122	1984	117-118	All matter tends to form, tends to crystallize. Crystals are formed by the Spirits of Form out of the substance of the Spirits of Will.
[8. Dec.	1910]	Berlin	60	1983	166	Everything mineral is initially a solidification, then an ossification of the plant world.
[9. Feb.	1911]	Berlin	60	1983	333	The living came into existence before the lifeless.
28. March	1911	Prague	128	1991	153	A principle of form is active in every crystal.
30. Dec.	1911	Hanover	134	1990	72-75, 76ff.	Matter is a broken form, a ruined heap of spirit, broken spirit. Crystals are copies of spiritual forms. Splitting occurs in a void: mineral matter.
18. Jan.	1912	Berlin	61	1983	272-273	We can recognise a super-sensible formative principle in a crystal.
14. April	1912	Helsinki	136	1996	185-190, 198	Crystal forms may be traced back to the modes of working of the Spirits of Form; their substances may be traced back to life forces, originating in the planets. Group ego of minerals: outside the planetary system, Spirits of Will.

12. Feb.	1914	Berlin	63	1986	262-263	The laws inherent in a mountain crystal manifest themselves in the crystal world
15. August	1916	Dornach	170	1992	158	Crystallography – and of necessity mineralogy too – are sciences having their basis in reality.
22. Sept.	1918	Dornach	184	1983	178-180	The cosmic mineral kingdom is formed in a crystalline manner. Crystal forms are peculiar to the mineral kingdom. Human corpses work in a manner opposed to the crystallization tendency of minerals.
Note	1919		320	1987	After 192	Crystals result from complicated encounters of force; atoms from simpler ones. Amorphous masses arise from the neutralization of the directions of force. Force is the manifestation of spirit viewed in a one-sided spatial manner. We cannot say that force works upon matter, since matter only consists of the array of effects from the converging rays of force.
29. Dec.	1919	Stuttgart	320	1987	113	In a certain respect a single crystal is a whole, yet a cut flower is not.
6. March	1920	Stuttgart	321	1982	97-98	Under normal conditions solid bodies take on a crystal form. How does this occur?
7. March	1920	Stuttgart	321	1982	109-114	Spherical form as the synthesis of all polyhedral crystal forms. The spherical form is the transitional form when passing from solid to gaseous configurations. The latter are negative forms in relation to spherical and polyhedral forms. In the 'warmth-night' the earth possesses the tendency to crystallization.
8. March	1920	Stuttgart	321	1982	124-125	Whenever a shape emerges in the transition from the fluid to the solid state, new forces must enter. They cannot already be present in the fluid. Therefore a new creation occurs.
[9. March	1920]	Stuttgart	321	1982	136-140	In the solid realm formations occur that are closed within themselves. Polarization: formation works upon formation.
23. March	1920	Dornach	312	1985	70-71, 73	Iron is the only metal exhibiting an orderly crystallization ability when coming in contact with the human organism. Crystals exhibit degenerative forces of evolution, in contrast to the formative forces of plants and animals.
24. March	1920	Dornach	[73a]	[1950]	[20]*	Atom hypothesis, diffraction of x-rays in crystals.
29. March	1920	Dornach	312	1985	188	The diverse crystal forms of silicate, particularly quartz, exhibit a scattering element.
8. April	1920	Dornach	312	1985	353	Crystallization forces in antimony.
18. July	1920	Dornach	198	1984	297	The same forces that give the earth its form are active in crystals.
[18. March	1921]	Stuttgart	324	1991	48	Application of mathematics to the lifeless mineral world.
[21. March	1921]	Stuttgart	324	1991	86	Projective geometry penetrates into the inner configuration of spatial forms; first stage in the transition to Imaginative knowledge.
7. April	1921	Dornach	76	1977	158-163	Polyhedral crystal forms must be related to the external aspect of the cosmos. Crystal forms are impresses of cosmic relationships.
17. April	1921	Dornach	315	1981	81-83	The organs of egotistic people possess the tendency to become crystalline. Spiritual activity arouses crystallization forces.
1. July	1921	Dornach	205	1987	89	Organic substance consists of matter that has cast out the force of crystallization.
1. Sept.	1921	Stuttgart	78	1986	78	In the inorganic sciences one doesn't experience the plastic formation of the crystal, but calculates the angles of the faces etc.
2. Oct.	1921	Dornach	207	1990	90-91	We draw the geometric thought-forms of crystal

						systems from out of the web of thoughts incorporated into us at birth.
5. Nov.	1921	Dornach	208	1992	140-149	Lunar forces give rise to the polyhedral aspect of crystals forms, as well as in a definite sense to the polyhedral forms in the other kingdoms of Nature.
12. Dec.	1921	Dornach	209	1982	96	What is taught by crystallography, can scarcely be compared with what we experience when beholding a crystal.
9. August	1922	Dornach	347	1995	44-63	Crystal formation.
28. Oct.	1922	Stuttgart	314	1989	153	Rounded and oblong crystal forms.
13. Jan.	1923	Dornach	220	1982	78-82	Non-crystallized substances are disturbed crystallizations. 'Salt' process as the process of crystal formation. Crystallizations are the embodiments of individual words of the Universal Logos.
30. Nov.	1923	Dornach	232	1987	57-60, 66	Crystal covering of the earth. The crystalline element of the earth as its sense organ. Antimony crystals: follow certain effects of force in the cosmos.
1. Dec.	1923	Dornach	351	1988	166-170	Quartz crystal, silicic acid, the shape of honeycomb.
10. June	1924	Koberwitz	327	1984	49-50	The greatest crystallization force is developed by the earth from the middle of January to the middle of February.
21. July	1924	Arnheim	319	1994	172	Even a quartz crystal is not inert: the process merely occurs at an extremely slow rate.
13. August	1924	Torquay	243	1993	48-54, 54-62	The Being of crystallized minerals. Substantiality and metallicity in the mineral world. Interplay between form and substance.
[18. Sept.	1924]	Dornach	354	1977	195-211	Tetrahedral form of the earth.

* Rudolf Steiner, "Anthroposophie und gegenwärtige Wissenschaften" (24.3.1920). In: *Geisteswissenschaft und die Forderungen der Gegenwart* (lectures from the year 1920), Heft V (Dornach: Selbstverlag der Rudolf Steiner-Nachlassverwaltung).

B. Literature on the Theme of Crystallography from the Domain of the Anthroposophically Extended Natural Sciences.

The following list contains writings known to me from the domain of crystallography that are more or less closely related to researches having their basis in an anthroposophically extended view of natural science. Those works have been primarily selected that refer to studies on the morphology of individual crystals.

Correspondingly, writings on diagnostic copper-chloride crystallization are not generally included, since they mostly treat of the morphology of crystal aggregates in solutions containing organic substances (see further below). For a small charge a comprehensive list of relevant literature may be obtained from the 'Laboratorium für empfindliche Kristallisation' Goetheanum, CH-4143, Dornach 1, Switzerland.

A complete evaluation of these works cannot be carried out here. However, for most of them a few indications about their content are provided.

The works of *Adams* (1931, 1934) and *Stockmeyer* (1931) are fundamental to the view proposed here and in *Ziegler* (1998) concerning a geometric crystal morphology based on projective geometry. *Schwenk* (1961) and *Mackensen* (1988, p. 155f.) explicitly refer to the significance of these writings for a comprehensive understanding of crystal formation,

Teichmann (1981) implicitly refers to them. *Kötter* links on to the works of *Adams* and *Stockmeyer* in a manner similar to ours; also see *Bernhard* (1984). *Adams* (1957), *Halla* (1955) and *Ziegler* (1981) all discuss the significance of the reciprocal lattice.

An individual and original contribution on crystal morphology is furnished by *Arnoth* (1997). This exposition is currently being developed and shall appear in expanded form in the near future.

Vital (1976) examines the singular geometrical directions within the original area of emergence in an alpine mineral cleft, as well as the effects of gravitation.

A differentiated phenomenological work on the natural morphology of salt crystals and salt deposits in solid, dissolved and atmospheric formations may be found in *Kopp* (1974, 1978). The distinction between centrally and peripherally active forces included in the latter work does not bear any directly recognisable connection to the methodological approaches of *Adams* and *Stockmeyer*. Nonetheless, it is (besides *Adams*) the only work known to me going beyond a conventional presentation of crystal formative forces, and which also goes beyond mere generalities. *Göbel* (1981) partly takes up these ideas, but doesn't add anything significantly new.

Nitschmann (1980) discusses the form-metamorphoses in the growth of cooking-salt crystals; his (1984) work expands on these studies to include both other substances and general regeneration processes in crystals. Ultimately, in (1990) these processes are illuminated by cognitive results derived from the spiritual science of *Rudolf Steiner*, and further continued in (1992ab) in a somewhat speculative manner.

With pedagogical aims in mind, *Mackensen* (1977abc, 1988) provides a revised sketch of the traditional morphologically orientated theory of crystals, placing the phenomenological aspects in the foreground. In (1977b), his proposed phenomenological ordering of 'crystal families' solely on the basis of the multiplicity of symmetry axes isn't entirely convincing. This ordering is not rigorous enough for a systematic and thorough crystallography which takes geometrical-mathematical viewpoints seriously (especially the theory of symmetry groups). For the conception of the symmetry axis does not stand apart by itself, and cannot be viewed in isolation, but must of necessity be embedded in the entire symmetry relationship (group). On a basis such as this the classical division into the seven crystal systems may then be established without any direct relation to a co-ordinate system, which may seem unnatural from the phenomenological viewpoint (cf. *Ziegler* (1998), chapter 4.4, p. 60).

Benesch (1990), *Göbel* (1981), *Mackensen* (1977b), *Schad* (1981), and *Teichmann* (1981) all specifically refer to the complementary character of the shape (morphology) and substance (lattice-structure), without however examining the viewpoint of projective geometry underlying the morphology.

Benesch (1990), *Pfeiffer* (1926), *Schüpbach* (1997) and *Wimmenauer* (1992) discuss the chemical and physical properties in greater detail. *Edelglass* (1973) presents the conventional viewpoint with respect to the lattice.

The formation of crystals as a part-process of all-encompassing geological and historical processes, both on and in the earth, are understood in different ways in *Bosse/Goethe* (1994), as well as in the writings of *Cloos* (1951, 1956, 1958, 1966a), *Knauer* (1961), *Schmutz* (1986) and *Wimmenauer* (1994).

The origin of the mineral kingdom from out of the organic kingdom, as well as a comparison between life and mineral processes, may be found in the works of *Gehlig* (1993, 1994). Also see *Pfeiffer* (1921/22, 1926) and *Schornstein* (1939).

The works of *Ballivet* (1993), *Beckmann* (1959), *Goedings* (1983), *Keller* (1961, 1964), *Knauer* (1971), *Neuhaus* (1957, 1960), *Nickel* (1968), *Pfeiffer* (1927, 1930), and *Selawry* (1957) all concentrate on crystallization in shallow dishes. In regard to the morphology of individual crystals, reference is made to the known fact about the dependency of the form of a crystal, particularly to related solvents, and more generally to their chemical

and physical environment. *Keller* (1961) contains a number of related experimental investigations with sodium sulphate; the (1964) continuation of these studies reports on the relevant morphological influences of various types of light (sunlight, coloured light, starlight, artificial light).

In conclusion, I mention the next few works because they stand in a close relationship with the topic of the present paper, or seem especially valuable to me for more in-depth researches leading to a morphology on a projective basis. In addition to the works of *Adams* (1931, 1934), *Stockmeyer* (1931) and *Kötter* (1979), see the following: *Keller* (1964), *Kopp* (1974, 1978), *Nitschmann* (1980, 1984, 1990), *Schad* (1981) & *Schmutz* (1986).

References

- Adams, George* (1931): Synthetische Geometrie, goethesche Metamorphosenlehre und mathematische Physik. In: *Mathesis, Beiträge zur Weiterbildung der Mathematik und verwandter Gebiete im Sinne der Geisteswissenschaft*. Stuttgart, pp. 119-173.
- Adams, George* (1934): Die Kristallgestaltung des Raumes. In: *Strahlende Weltgestaltung. Synthetische Geometrie in geisteswissenschaftlicher Beleuchtung*. Dornach, Mathematisch-Astronomische Sektion am Goetheanum. (2nd ed. Dornach 1965), chapter 7, pp. 325-386.
- Adams, George* (1957): Das 'reziproke Gitter' und die Röntgenanalyse der Kristalle. *Mathematisch-Physikalische Korrespondenz*, No. 12, p. 48. Reprinted in: *G. Adams* (1996): *Universalkräfte in der Mechanik*. Dornach, Mathematisch-Astronomische Blätter (Neue Folge, volume 20), pp. 213-224.
- Arnoth, Joseph* (1997): Kristallform (Kristallgestaltung. Begleittexte zur Sonderausstellung 24. October 1997 (28. June 1998 (Manuscript from Joseph Arnoth with a contribution from Renatus Ziegler), Naturhistorisches Museum, Basel. (2nd expanded edition 1998).
- Ballivet, Christine* (1993): Empfindliche Kristalle zwischen unorganischer und organischer Natur. *Elemente der Naturwissenschaft*, No. 58, 1993, Heft 1, pp. 76-82.
- Beckmann, H.* (1959): Über Keimbildung, Einkristallwachstum und Auffächerungswachstum von $\text{CuCl}_2 \cdot 2\text{H}_2\text{O}$ in rein wässrigen eiweißhaltigen Lösungen. Dissertation, Universität Bonn 1959.
- Benesch, Friedrich* (1981): Die Apokalypse. Die Verwandlung der Erde. Eine okkulte Mineralogie. Stuttgart.
- Benesch, Friedrich* (1990): Der Turmalin. Eine Monographie. Stuttgart.
- Bernhard, Arnold* (1984): Dualität im Raum – vollständiges Hexaeder und vollständiges Oktaeder – zentriert-planisierte Bewegung. In: *Projektive Geometrie – aus der Raumanschauung zeichnend entwickelt*, Stuttgart (Menschenkunde und Erziehung, vol. 45), Chap. 21, pp. 105-114.
- Bosse, Dankmar / Goethe, Johann Wolfgang* (1994): Die Gestalten der Kristalle. In: *Johann Wolfgang Goethe, Die Metamorphose des Granits, Substanz und Gestaltbildung des Erdorganismus* (zusammengestellt und bearbeitet von Dankmar Bosse), Stuttgart, pp. 279-295.
- Bühler, Walther* (1996): Quasi-Kristallbildung macht das Unmögliche möglich. In: *Das Pentagramm und der goldene Schnitt als Schöpfungsprinzip*. Stuttgart, pp. 354-370.
- Cloos, Walther* (1952): Die Erde – Ein Lebewesen. Beiträge zur Physiologie der Erde. Stuttgart.
- Cloos, Walther* (1956): Kleine Edelsteinkunde im Hinblick auf die Geschichte der Erde. Freiburg i. Br.
- Cloos, Walther* (1958): Lebensstufen der Erde. Beiträge zu einer organischen Gesteins- und Mineralkunde. Freiburg i. Br.
- Cloos, Walther* (1966a): *Werdende Natur*. Dornach.
- Cloos, Walther* (1966b): Der Salz-Prozess in der Natur und im Laboratorium. In: *Cloos* (1966a), pp. 118-130.
- Edelglass, Stephen* (1973): Crystallinity, Form, and Symmetry. *Mathematical-Physical Correspondence*, no. 6, pp. 2-7.
- Gehlig, Rosemarie* (1993): Von der Eigenart der Bildekräfte im Mineralreich. In: *Tycho de Brahe-Jahrbuch für Goetheanismus 1993*, pp. 14-63.

- Gehlig, Rosemarie* (1994): Kiesel, Kalium, Calcium und Wasser als Leitelemente in den Naturreichen. Ein Bild von Lebensvorgängen in der Gesteinswelt. In: Tycho de Brahe-Jahrbuch für Goetheanismus 1994, pp. 131-150.
- Göbel, Thomas* (1981): Metamorphosen im Mineral- und Pflanzenreich: Gestaltwandlung im Mineralreich; Gestaltwandlung im Pflanzenreich. Die Drei, Jg. 51, Heft 7/8, pp. 564-570, 571-575.
- Goedings, Peter* (1983): Die Dynamik der empfindlichen Kristallisationsmethode in den Lebensprozessen des Menschen. Beiträge zu einer Erweiterung der Heilkunst nach geisteswissenschaftlichen Erkenntnissen, 36. Jg., 1983, Heft 3, pp. 85-101.
- Halla, Franz* (1955): Raum und Gegenraum. Mathematisch-Physikalische Korrespondenz, no. 4, pp. 4-5.
- Keller, Friedrich Otto* (1961): Kristallmorphologie und Umwelteinflüsse. Jahrbuch des Museums Hohenleuben-Reichenfels (Hohenleuben/Zeulenroda, Thüringen), Heft 10, pp. 120-164.
- Keller, Friedrich Otto* (1964): Licht formt Kristalle. Jahrbuch des Museums Hohenleuben-Reichenfels (Hohenleuben/Zeulenroda, Thüringen), Heft 12/13, pp. 144-171.
- Keller, Friedrich Otto* [under the pseudonym: Friedrich Okel] (1965): Formschaffende Gestaltungskräfte und ihr qualitativer Nachweis. Die Kommenden, Jg. 19, no. 10, pp. 18-20.
- Knauer, Helmut* (1961): Erdenantlitz und Erdenstoffe: Gestaltung und Aufbau des Erdorganismus. Dornach.
- Knauer, Helmut* (1971): Aus dem Wesen der Stoffeswelt. Kristallisation als Ausdruck von Bildekräften der mineralischen Welt. Die Drei, Jg. 41, Heft 9, pp. 416-422.
- Kötter, Dieter* (1979): Die Phänomene der Kristallformen im Lichte der synthetischen projektiven Geometrie. (Projektarbeit, Naturwissenschaftliches Studienjahr, Dornach 1978/79; unpublished manuscript.)
- Kopp, Werner* (1974): Die Salze im Naturzusammenhang. In: H. Kaufmann, H. B. von Laue (eds.), Der Homöopathisierungsprozess bei Rudolf Steiner, Tagung 1974 (Weleda Schriftenreihe, vol. 8), Autorreferate der Tagungsbeiträge, pp. 15-29.
- Kopp, Werner* (1978): Kristallisationsprozess und Verwandlung der Substanzzustände der Salze als Verständigungsgrundlage für die Mineralisierungsprozesse im Menschen. In: Bruno Busse / Roland Schaette (eds.) (1978): Der Homöopathisierungsbegriff bei Rudolf Steiner, Referate der Tagung 1977 der Anthroposophisch-Pharmazeutische Arbeitsgemeinschaft. Stuttgart, pp. 34-48.
- Kügelgen, Angelika von / Schad, Wolfgang* (1973): Vom Kristalle-Züchten. Erziehungskunst, Jg. 37, Heft 5, pp. 212-222.
- Locher, Louis* (1957): Die Fundamentalstruktur. In: Raum und Gegenraum – Einführung in die neuere Geometrie, Dornach, Chap. I.5, pp. 29-36.
- Mackensen, Manfred von* (1977a): Symmetrien, Kristallklassen, Mineralien (I): 1. Symmetrie in verschiedenen Weltbereichen; 2. Symmetrieelemente von Kristallen. Erziehungskunst, Jg. 41, Heft 3, pp. 142-151. Reprinted in slightly modified form in: (1988), pp. 35-40, 128-135.
- Mackensen, Manfred von* (1977b): Symmetrien, Kristallklassen, Mineralien (II): 3. Symmetriefamilien; 4. Phänomenologie und Gitterstruktur. Erziehungskunst, Jg. 41, Heft 4, pp. 198-205. Reprinted in slightly modified form in: (1988), pp. 135-145.
- Mackensen, Manfred von* (1977c): Symmetrien, Kristallklassen, Mineralien (III): 5. Einige häufig vorkommende Wachstumsformen. Erziehungskunst, Jg. 41, Heft 5, pp. 258-265. Reprinted in slightly modified form in: (1988), pp. 145-156.
- Mackensen, Manfred von* (1988): Salze – Säuren – Laugen. Phänomenologische Chemie für den Unterricht in der 10. Klasse. Kassel (2nd ed.).
- Maile, Walter* (1984): Pyrit-Kristalle. Mitteilungen aus der anthroposophischen Arbeit in Deutschland, no. 148, 8. Jg., Heft 2, pp. 111-115.
- Neuhaus, A.* (1957): Die kristallographischen Grundlagen der ‘Diagnostischen Kristallisationen’. In: *Selawry, Alla and Oleg* (1957), pp. XIV-XL.
- Neuhaus, A.* (1960): Was sagt die Wissenschaft zur Kupferchlorid-Diagnostik? Ein Beitrag zum Problem der diagnostischen Kristallisationen. Die Umschau in Wissenschaft und Technik, Heft 16, 1960, pp. 486-489.
- Nickel, Erwin* (1968): Die Reproduzierbarkeit der so genannten ‘empfindlichen Kupferchloridkristallisation’. Bulletin de la société fribourgeoise des sciences naturelles, vol. 57, 1967/68, issue 2, pp. 65-179.

- Nitschmann, Günter* (1980): Ein offenes Geheimnis. Die Metamorphose beim Wachstum von Kristallkugeln. *Die Drei*, Jg. 50, Heft 10, pp. 617-623.
- Nitschmann, Günter* (1984): Die Überschusskräfte des Mineralreiches und die Regeneration von Kristallen. *Elemente der Naturwissenschaft*, no. 40, 1984, Heft 1, pp. 35-41.
- Nitschmann, Günter* (1990): Welches 'Selbst' organisiert die Kristallisation? *Das Goetheanum*, Jg. 69, no. 48, pp. 429-430.
- Nitschmann, Günter* (1992a): Die vierfache Organisation des Kristallkörpers. Referat bei dem 4. Arbeitstreffen der Arbeitsgruppe 'Kosmologie' der Naturwissenschaftlichen und der Mathematisch-Astronomischen Sektion der Freien Hochschule für Geisteswissenschaft am Goetheanum in Berlin.
- Nitschmann, Günter* (1992b): Der Kristall – ein trinitarisches Wesen. *Das Goetheanum*, Jg. 71, no. 3, pp. 24-25.
- Nitschmann, Günter* (1993): Gedanken zur empfindlichen Kristallisation mit Kupferchlorid. *Elemente der Naturwissenschaft*, no. 58, 1993, Heft 1, pp. 14-19.
- Pfeiffer, Ehrenfried* (1921/22): 'Kristallseelen'. *Das Goetheanum*, Jg. 1, no. 35, pp. 280-282.
- Pfeiffer, Ehrenfried* (1926): Altes und Neues zur Erkenntnis des Mineralischen im Hinblick auf die Heilkunst. *Natura*, vol. 1, pp. 187-194.
- Pfeiffer, Ehrenfried* (1927): Aus dem naturwissenschaftlichen Forschungslaboratorium in Dornach. In: *Gäa-Sophia – Jahrbuch der Naturwissenschaftlichen Sektion der Freien Hochschule für Geisteswissenschaft am Goetheanum Dornach*, vol. II, pp. 89-105.
- Pfeiffer, Ehrenfried* (1930). Kristalle. Stuttgart (Berichte aus den Arbeiten des Naturwissenschaftlichen Forschungslaboratoriums am Goetheanum, Dornach.)
- Schad, Wolfgang* (1981): Bergkristalle – Thema mit Variationen. *Erziehungskunst*, Jg. 45, Heft 1, pp. 5-17.
- Schmutz, Hans-Ulrich* (1986): Die Tetraederstruktur der Erde. Eine geologisch-geometrische Untersuchung anhand der Plattentektonik. Stuttgart.
- Schornstein, Waldemar* (1939): Kristalle. In: *Alfred Erhardt*, Kristalle; 64 Aufnahmen von Alfred Erhardt; Mit einer Einführung von Dr. Waldemar Schornstein (Hamburg: Heinrich Ellermann), pp. 5-13.
- Schüpbach, Martin* (1997): Kristalle. Licht und Farben im Stein. Dornach.
- Schwenk, Theodor* (1961): Kristallbildung aus dem Umkreis. In: *Schmidt, Gerhard et. al.* (eds.), Heilmittel für typische Krankheiten, gegeben durch Dr. Rudolf Steiner. Vol. III: Antimon (Dornach: Medizinische Sektion am Goetheanum), pp. 73-83.
- Selawry, Alla and Oleg* (1957): Die Kupferchlorid-Kristallisation in Naturwissenschaft und Medizin. Stuttgart.
- Stockmeyer, Ernst A. K.* (1931): Ein Versuch über die Universalkräfte in der Kristallgestaltung. In: *Mathesis* (1931), Beiträge zur Weiterbildung der Mathematik und verwandter Gebiete im Sinne der Geisteswissenschaft. Stuttgart, pp. 241-260.
- Teichmann, Frank* (1981): Die Gestalt der Kristalle: Gesetz und Erscheinungsbild. *Die Drei*, Jg. 51, Heft 7/8, pp. 557-563.
- Vital, Annina* (1976): Eine alpine Miniaturkluft aus der Lukmanierschlucht. *Mathematisch-Physikalische Korrespondenz*, no. 100, pp. 25-31.
- Wimmenauer, Wolfhard* (1992): Zwischen Feuer und Wasser. Gestalten und Prozesse im Mineralreich. Stuttgart.
- Wimmenauer, Wolfhard* (1994): Der Granit – 200 Jahre nach Goethe. In: *Tycho de Brahe-Jahrbuch für Goetheanismus* 1994, pp. 93-129.
- Wimmenauer, Wolfhard* (1995): Kugelformen im Mineralreich. In: *Tycho de Brahe-Jahrbuch für Goetheanismus* 1995, pp. 39-65.
- Ziegler, Renatus* (1981): Untersuchungen zur Mathematischen Kristallographie. Zürich (Diplomarbeit ETH; unpublished typescript).
- Ziegler, Renatus* (1998): Morphologie von Kristallformen und symmetrischen Polyedern. Kristall- und Polyedergeometrie im Lichte von Symmetrietheorie und projektiver Geometrie. Dornach, *Mathematisch-Astronomische Blätter*, Neue Folge, vol. 21.

Acknowledgements:

Arnold Bernhard, Dankmar Bosse, Christoph Hennig, Albert Pröbstl, David Speiser and Wolfhard Wimmenauer all read the manuscript and commented upon it in various ways. Chapter 7 was conceived as a result of stimulating conversations with Georg Maier. I'm especially grateful to Helmut Fischmeister for his untiring and constructive criticisms from the standpoint of a traditional physicist; his suggestions greatly helped me to more clearly formulate my position. I also wish to thank Christine Ballivet and Damian Quero for their assistance in the research of the literature. And special thanks go to David W. Wood for his fine and competent translation.

Dr. Renatus Ziegler
Mathematical-Astronomical Section
Goetheanum
Ch-4143 Dornach 1
Switzerland

Translated from the German by David W. Wood
