Introduction

Finsler's attitude towards mathematics was Platonistic in a very definite sense: He believed in the reality of pure concepts. Together they form the purely conceptual realm which encompasses all mathematical objects, structures and patterns. This realm exists independently of any particular state of human consciousness or individual experience. Mathematicians do not invent or construct their structures and propositions; they recognize, or discover, how these objects in the conceptual realm are interrelated with each other.

It is clear that if there exists a conceptual realm, then it must be absolutely consistent; hence *existence implies consistency*. This implication, however, does not suffice to prove the existence of pure concepts. The *Platonistic perspective* of mathematics can be expressed by the converse implication: *Consistency implies existence*. If a concept has been found to be consistent, it can be assumed to exist. This means that one can find properties and prove theorems about it.

At first, it may seem unnecessary to ask whether a conceptual entity is real, or has an ontological status beyond its consistency. Exactly this question, however, was at stake during the foundational crisis. Many critical thinkers contended that it is precisely this naive notion of existence that lies at the heart of foundational problems. Lacking a consistent and convincing Platonist philosophy, Hilbert, and many other mathematicians and logicians along with him, required that a mathematical object must be expressible in some language to really exist. Hilbert's approach to foundations tied mathematical existence to symbolic representation, that is to linguistic expressions in a strictly formal language.

Finsler entered the debate at this very point. He maintained that consistency is sufficient for the existence of mathematical objects. Furthermore, he thought that the antinomies which led to the foundational crisis, could be solved without the notion that existence is equivalent to formal constructibility. His main intention, visible in his writings, was to go back to the very roots of a strictly Platonistic interpretation of mathematics as in Cantor's set theory. Hence, Finsler's thoughts require a re-examination of basic issues in the philosophy of mathematics that are still unsolved, or at least have solutions that are not universally accepted.

Contrary to Cantor, however, Finsler never discussed his philosophical perspectives at any length. He assumed them to be self-explanatory for working mathematicians, that is, he assumed them to be clear from their experience. This might have been the case for most mathematicians in the 19^{th} century but certainly not for the critical way of thinking which emerged from the foundational crisis.

In this introduction there will be a short reconstruction of Finsler's philosophy of mathematics. This is still a largely unexplored territory with many open problems. However, it is indispensable for an understanding of his purely mathematical research. It can be shown that Finsler's point of view is at the very least internally consistent, that is, an hypothesis which has to be taken seriously. Apart from that, it is inspiring and may start some fruitful future research.

One of the most important distinctions for a Platonist outlook on mathematics is the one between pure concepts and their verbal or symbolic representation. The latter is no substitute for the former: An expression merely points to the structure or pattern that it refers to. The pure concept is accessible to someone who makes the effort to *think* what is meant by such a linguistic expression. In particular, the notions of truth and consistency have their primary meaning beyond language: Their structure is, in the first place, not a matter of linguistic distinctions (for example between object language and metalanguage) but of *understanding* or *insight*.

It may be an easy matter to change the notation in which a theorem or a definition is expressed. We often translate – mathematicians are well accustomed to it – a theorem into some other language. The theorem itself, however, to which these notations or formulations refer, is invariant under merely linguistic transformations. The theorem itself cannot be altered. It is something which we become distinctly aware of as soon as we really *think* about it.

The realm of pure concepts is accessible by *insight*, or pure thought (some mathematicians – for example Gödel [1964] – call this "mathematical intuition"). This is part of the everyday experience of a mathematician, not something mystical. He might prefer to call it *informal thinking*, or more appropriately *nonformal*, instead. He experiences it above all in those moments in which he is not simply manipulating symbols or in which his thoughts have not yet been symbolically expressed. In particular, logical calculi, like all calculi, are manipulative (done by hand or machine) so they cannot capture thinking, though they may reflect it.

What is meant by "informal thinking"? There is a whole range of conceptual qualities to which this might refer: From a well thought out theory to a very vague, or even highly speculative, conjecture. These individually conceived conceptual structures all have in common this *nonformal* nature.

When a mathematician has an idea which gives a new insight and is important for his topic of research, then he tries, at least in principle, to organize his thoughts into a rigorous, deductive pattern of arguments which represents the original nonformal insight as closely as possible. This process, although it might include writing or symbol manipulation, is conceptual in its essence. Even if one goes as far as expressing one's ideas within a strictly formal language, the main goal is still to represent the initial idea adequately. Such procedures are reasonable ones; it may appear, however, as if the clarity and rigor of the final structure is due to its formal style of presentation. But how was the clarity and security of the intended informal thought patterns achieved? What are the criteria by which a mathematician judges the process and final result of the formal in comparison with the preformal stage? It is by his or her own nonformal insight, or understanding, which started and accompanied the whole process of formalization. This shows that the nonformal insight is prior, systematically and temporally speaking, to the formal one. We would not know what to formalize if it were not for the nonformal insight, the pure concept that we are aware of. *Formalization occurs at the end and not at the beginning of the true path to mathematical understanding.*

An opponent of Platonism might argue that preformal insight is vague by its very nature and hence cannot really be the source for any precise mathematics. Platonists, however, do not argue against organizing the initial fuzzy thoughts or intuitions into rigorous chains of arguments based on a set of clearly defined assumptions or axioms. Furthermore, they are aware of the fact that in writing something down, one increasingly clarifies the ideas. But, independently of how far one goes in spelling out the details symbolically, it is still the nonformal insight which guides writing and not the syntactic rules of language.

Consequently, the final linguistic expression is a mere *representation* of the real thought – and is not to be confused with this thought, or concept, itself. If a symbolic expression is given, one usually refers to the corresponding thought or concept as its *meaning* or *content*.

Let us come back to our primary distinction between conceptual content and linguistic expression. From this point of view, the distinction between *object language* and *metalanguage* (or mathematics and metamathematics) presents itself as a projection of the former distinction onto the realm of language. Without the former distinction, the latter would be artificial. This becomes evident if one proceeds to formalize the metalanguage itself. In this case, assertions of truth, meaning etc. about an expression in the given formal language (object language) appear just as another string of symbols which in itself do not explain themselves but would need a meta-metalanguage. In practice one usually uses natural language as a metalanguage (including the meta-metalanguage etc.) of the given object language. However, natural language is only another means of expression which bears neither truth nor meaning in itself but asks for some non-linguistic, i.e. conceptual interpretation. Effectively, truth and meaning can only be found in the pure conceptual realm if one does not want to fall into an infinite regress, that is, an open-ended hierarchy of languages.

When a Platonist like Finsler refers to a theory, to mathematical objects, or to a set of axioms, he refers to objects in the purely conceptual realm. The specification of a formal *language* has no part in his purely *mathematical* deliberations. *Mathematics is concerned with relationships between concepts and not with the expression of concepts in language.*

Of course, a Platonist also represents his thoughts with the help of languages, but he is well aware of the fact that it is his insight which gives meaning to his words and not the other way round.

What then, one might ask a Platonist, is the function of language? Why use it at all?

The primary purposes of language in mathematics are *communication, symbolic computation,* and *checking.* As far as mathematical *insight* goes, there is, strictly speaking, no need for a language. *Mathematics is not the science of communication of structures and patterns, but the science of these structures and patterns themselves.* However, if one wants to tell someone else about one's discoveries, there is no way around using some kind of language to express them. In addition, it is helpful for storing one's thoughts (in the form of their symbolic representations) in an external memory, or for checking the results by some well-known computational methods.

As for *symbolic computation,* the need of appropriate notations for accurate and efficient symbol manipulations is evident. However, the meaning of the symbols and the rules of computation do not emerge merely from the rules of the syntax nor the grammar of the relevant language. Calculations are based upon a set of rules that are implemented in language from a realm outside of it. Thus, the results of computation need to be interpreted, apart from merely symbolic checks.

However, if it were not for communication or symbolic computation, there would be no necessity for language; mathematical insight would still be there without any language. Mathematicians might write down their ideas or compute something symbolically in order to *check* the results against some prior knowledge or with acknowledged, secure methods. However, they do not need to write their thoughts down in order to *understand* them. If this were the case in its strictest sense, how could they ever know the meaning of what they wrote down?

We need to be careful not to confuse the complicated and sometimes rather "irrational" search for conceptual clarity, during which we might go through different stages of computing, writing and editing, and the purity of insight that we arrive at in the end. We are only concerned here with the latter: The final clear insight. It transcends the symbolic patterns, as everyone knows who tries to write or read *and* understand a mathematical paper; it is not enough to recognize symbols, to know their syntactical structure, or to be able to follow the pattern of a symbolic computation. One needs to think and thus grasp the meaning of the thoughts which are to be communicated.

From this point of view it should be clear that most Platonists are not interested in the fine structure of a language for its own sake, but only as a means of expressing pure thoughts. What they want to understand are the concepts themselves, not just verbal or symbolic representations.

Let us now turn to Finsler's philosophical papers from the standpoint of an historian, taking leave of the Platonist point of view.

Are there Contradictions in Mathematics? [1925]

This paper is a preview of Finsler's future research on the foundations of mathematics, set theory in particular. To begin with, he announces his intention to restore the consistency of mathematics by solving, not avoiding, the antinomies. One does not need a new logic nor a correction of the old one for this purpose.

In dealing with what Finsler calls "logical antinomies" (later called "semantical antinomies"), i.e. the antinomies of the "liar" and the antinomy of finite definability, he introduces the distinction between the explicit and the implicit content of a proposition. The "explicit content" refers to the conceptual meaning and the "implicit content" to the form of representation. Antinomies arise if these two "contents" contradict each other.

Concerning the "set theoretic antinomies", in particular Russell's paradox, Finsler points out that one needs to distinguish between satisfiable and unsatisfiable circular definitions. Russell's definition of the set of all sets which do not contain themselves is a non-satisfiable circular definition. Finsler, however, maintains that it is not necessary to exclude all circular definitions because of that; they are used even in algebraic equations.

Solving the antinomies does not positively solve the problem of a consistent foundation of set theory. That task is reserved for the paper *On the Foundations of Set Theory* [1926b] in Part II of this book.

Formal Proofs and Decidability [1926a]

In this paper Finsler establishes the formal undecidability of a proposition which is, however, false. From this he concludes that formal consistency does not imply absolute consistency.

In carrying out his proof, Finsler is not so much concerned with a precise definition of a formal system as with the demonstration of the limitation of any kind of symbolic representation. In order to show that there *are* formally undecidable propositions, he refers to the fact that any language uses at most countably many symbols. Hence, not all propositions of the form:

α is a transcendental number,

are expressible, or definable in a language, since there are uncountably many transcendental numbers. Consequently, since only countably many of these propositions can be formally proved within the given language, there must exist propositions of this kind which cannot be proved in these terms but which are still true.

Finsler goes on to present an example of a proposition that is formally undecidable yet false. In order to show the latter, he refers to the conceptual content of the verbal expression in question. He shows that if this conceptual content is taken into account, then the formally undecidable proposition turns out to be false.

One might summarize the argument here as follows: If there is a purely conceptual realm, no formal representation can capture it.

In effect, Finsler's main intention is not to distinguish between different kinds of formal systems but between the purely conceptual realm and its symbolic ("formal") representation, including the use of natural language. This is why he did not need to specify more precisely his notions of formal proofs, formal definability, formal systems etc.; every thing which is written down is formal in Finsler's sense. Hence, for his purpose, there is no need of a general reconstruction of language.

From this, the comparison between Finsler's incompleteness argument and Gödel's incompleteness proof [1931] takes on a new perspective. There are indeed striking similarities between Finsler's and Gödel's approach. However, as Van Heijenoort [1967] remarks in his introduction to Finsler's paper,

> Finsler's conception of formal provability is so profoundly different from Gödel's that the affinity between the two papers should not be exaggerated. [1967, 438]

This is certainly true, since Gödel's most profound achievements lie in the accurate definition of the particular formal systems in question and the concept of a formal proof within this system. Furthermore, he developed what is now called "the arithmetization of metamathematics"; for this purpose he gave a precise definition of the class of recursive functions. By precisely defining his formal methods, he shows, by constructing an example of an undecidable statement, that these formal methods are incomplete. An additional metamathematical argument then shows that this proposition which states its own unprovability is, in fact, true, and hence decidable on the metamathematical level. By these means Gödel achieved something that Finsler had not done: He proved even for the strictest formalist that formal means have their limits (see Dawson [1984] for further elaborations on this point).

Solely from the point of view of mathematics and formal logic, Gödel's paper is far more significant than Finsler's. However, Finsler's paper goes directly to the heart of the philosophical problem. Finsler is concerned with the fundamental distinction between concepts and their symbolic or verbal representation, not with the formally more sophisticated but philosophically limited distinction between metalanguage and object language. One might say that the latter distinction is the projection of the former distinction onto the realm of language.

Gödel was acutely aware of the objective Platonist principles behind the distinction between mathematics and metamathematics. Later in his life, he expressed strongly Platonist convictions, for example in the essays *Russell's mathematical logic* [1944] and *What is Cantor's continuum hypothesis* [1964], although he never exhibited these in his earlier writings on the foundations of logic. Fefermann [1988] argues that Gödel's extreme caution towards the power of formalist views of his time urged him to shy away from expressing his Platonist convictions until the Forties.

The audience Gödel wanted to address consisted of strict formalists. Their opinions were the only ones that mattered to him. This is why he restricted his analysis to the concepts and methods *they* could accept, namely, semantic distinctions, syntactic forms, restrictions to particular formal systems, and relative rather than absolute consistency. From the perspective of the strict formalist, apparently, what Finsler has done is "nonsensical" (see the quotation of Gödel in Dawson [1984, 82f.]), since it presupposes something the formalists reject: the existence of the purely conceptual realm. For instance, J.C. Webb thinks that the main achievement of the mechanization of Finsler's argument by Gödel was to bring "Finsler's undecidable sentences down from [the] "rein Gedankliche[n]" and put them back into the formal system. In short, he formalized Finsler's diagonal argument." [1980, 193]. Hence, in his opinion, there is no threat to mechanist or finitist convictions any more: There is nothing left a machine could *not* do. Webb misses the point, predictably however for a formalist of his persuasion, that this projection is not possible without severe philosophical effects, as was shown above.

It is not appropriate, however, to judge Finsler from this formalist point of view, even though he himself sometimes thought so (cf. Dawson [1984, 81]). Finsler wanted to prove that Platonism is a consistent and fruitful philosophical perspective (cf. Finsler [1941a]), by developing foundations for set theory in [1926b]. As a consequence of the arguments in his paper on *Formal Proofs and Decidability*, he could not accept any kind of formal restrictions concerning set theory, because set theory lies at the very heart of the *foundations* of mathematics itself. No formalized theory can ever capture foundational conceptions that bear upon *all* of mathematics. To use geometrical terms, formal theories apply only to local structures, not to global ones.

In concluding, it is important to note that no strict formalist will ever be convinced by Finsler's paper on *Formal Proofs and Decidability* [1926a], because Finsler assumes something that formalists cannot accept: the reality of pure concepts. Finsler did not make clear what he meant by that; this certainly limits the significance of his paper. However, we cannot exclude the possibility that the open questions about the consistency of the Platonist perspective of mathematics and the ontological status of the realm of pure concepts may be solved some day. Even Gödel [1964] could not say more than Finsler concerning his belief in the objective existence of the objects of mathematical intuition. Gödel chose not to refer explicitly to the reality of concepts in his purely mathematical research, whereas Finsler boldly did so.

On the Solution of Paradoxes [1927b]

In this paper Finsler expands on his ideas in the paper [1925] concerning the solution of paradoxes which involve circular definitions.

Are there Undecidable Statements? [1944]

Here Finsler compares his approach to incompleteness with Gödel's. His arguments are closely related to the "liar". He begins with a discussion of this paradox. What then follows is one of Finsler's most original contributions to the analysis of the semantic paradoxes. Finsler shows that there is an absolutely consistent proposition and that there is a statement which an individual mind cannot prove yet has to believe.

Some choose to call this paper "obvious nonsense" or even "almost pathological" without further elaborations (see Dawson [1984, 83]). We hope that this translation makes Finsler's arguments more accessible and less subject to misunderstandings.

The paper begins with the fundamental distinction between "formal" and "inhaltlich"; this is the distinction between formal representation within a symbolic language (or linguistic expression in general) and conceptual content. This distinction is instrumental in proving that there are, in principle, undecidable statements relative to a particular formal system which are nevertheless decidable conceptually, that is, decidable in an absolute sense.

Finsler maintains that it might superficially appear that Gödel referred to the conceptual realm when he showed, through a metamathematical argument, that there is a statement unprovable within the formal system which is decidable in the metasystem. However, if one takes into account that the metasystem can also be formalized, Gödel's incompleteness result only shows that there are undecidable statements *relative* to a given formal system. Such a system can always be enlarged in order to make the statement in question decidable. But then there will be another undecidable statement in this larger system and so on.

In view of this, Finsler argues that Gödel did not prove the existence of a proposition which is formally undecidable in principle. Finsler maintains that if one strictly *requires*, as a matter of principle, the formalization of all arguments involved (including the metamathematical ones), then Gödel's result becomes contradictory: The formally undecidable statement becomes formally decidable. This contradiction only disappears if one explicitly takes into account the conceptual realm, or if one severely restricts the available logical tools on the object level.

One might object that Finsler's arguments are only correct if one ignores the distinction between the object level and the metalevel. Indeed, this distinction is one of the major achievements of modern mathematical logic. However, Finsler never refers to it. Was he not aware of it? Or did he simply ignore it?

In fact, this distinction is of minor importance within Finsler's approach. He was not interested in consistency, completeness, decidability, etc. *relative* to a certain formal system, but in *absolute* consistency, in short, in *absolute* results. Consequently, he was not interested in studying the subtle effects of modifications, restrictions, or extensions of various formal systems, but in the analysis of the effects of formal representation itself. Hence there was no need for him to distinguish between the object language and the meta-language. This distinction exists only for concepts that are expressed in language.

Gödel's unpublished remark that Finsler's aim, to achieve absolute results, is "nonsensical" was at the very least hasty. After all, Gödel himself refered from time to time to absolute notions (see Dawson [1984] and Fefermann [1988]).

This paper on absolute decidability shows clearly what Finsler wanted: that mathematical thinking not artificially limit itself by requiring that formalization be an *essential* part of mathematical existence.

The discussion of the "liar" in §2 is based upon the distinction between the explicit conceptual content of a proposition and its implicit assertion that it be true or false. The paradox arises out of the fact that the implicit assertion which contradicts the explicit assertion, is ignored.

In §3 Finsler expands the notion of proof so that it includes all possible ideal proofs. He can then show that the assumption that there are no undecidable statements (i.e. no unsolvable mathematical problems) is absolutely consistent. In particular, he shows that it is impossible to prove that a certain proposition is absolutely undecidable. From this he deduces, in §4, one of his most original results: There is a statement, which I, myself, cannot prove yet need to believe, because it can be proved rigorously by someone else.

The Platonistic Standpoint in Mathematics [1956a]

This paper records Finsler's part of a discussion of foundational issues in the journal *Dialectica*. It contains a reference to Specker's objection which is treated in sections VII and VIII of the introduction to the *Foundational Part* of this book.

Platonism After All [1956b]

As in the paper above, this is Finsler's part of a discussion, not all of which is included here (see Wittenberg [1956], Bernays [1956], Lorenzen [1956]). It contains mention of Ackermann's set theory [1956], which is described in section X of the introduction to the *Foundational Part* of this book.

Finsler Set Theory: Platonism and Circularity

Translation of Paul Finsler's papers on set theory with introductory comments

Edited by David Booth
and Renatus Ziegler

Birkhäuser Verlag
Basel · Boston · Berlin

Editors' Addresses

David Booth
Three Fold Foundation 307
Hungry Hollow Road
Chestnut Ridge
New York 10977
USA

Renatus Ziegler
Mathematisch-Astronomische Sektion am Goetheanum
4143 Dornach
Switzerland

Mathematics Subject Classification (1991): 03A05, 03E05, 03E20, 03E30, 04A20

A CIP catalogue record for this book is available from the Library of Congress, Washington D.C., USA

Deutsche Bibliothek Cataloging-in-Publication Data

Finsler set theory: Platonism and circularity : translation of
Paul Finsler's papers on set theory with introductory comments
/ ed. by David Booth and Renatus Ziegler. - Basel ; Boston ;
Berlin : Birkhäuser, 1996
ISBN 3-7643-5400-3 (Basel...)
ISBN o-8176-5400-3 (Boston)
NE: B ooth, David [Hrsg.]; Finsler, Paul

Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Cover design: Markus Etterich, Basel
Cover illustration: Photo of Paul Finsler by courtesy of Regula Lips-Finsler, handwritten manuscript of Paul Finsler by courtesy of the Mathematical Institute, University at Zürich
Printed in Germany
ISBN 3-7643-5400-3
ISBN 0-8176-5400-3

9 8 7 6 5 4 3 2 1

Contents