Selected Topics in Three-Dimensional Synthetic Projective Geometry
Chapter 18

Fundamental Complexes

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There is a rich interweaving of two different types of duality within the five-dimensional manifold of linear complexes. The first type lies in three-dimensional projective space and can, for example, be realised in the polarity relative to a second degree surface. The second type, polarity on the universal line quadric, belongs in the five-dimensional manifold of linear complexes itself (see Sections 17.4.2 and 17.4.3).

Both, the polarity on a surface of second degree and the polarity on the universal line quadric are projectivities, specifically involutory correlations. This will not be proved here for the polarity on the universal line quadric in the five-dimensional manifold of linear complexes.

There arises a question concerning common invariant figures of both polarities, the so called *simultaneous invariants*. Such an invariant figure must be simultaneously invariant under both polarities. This issue will be approached in Section 18.1, and a particular invariant system of six complexes, the so-called *fundamental complexes*, will be studied in greater detail in Section 18.2.

18.1 Simultaneous invariants of two polarities

The question of common invariants of a polarity on a quadratic surface in three-dimensional projective space and a polarity on the universal line quadric is asymmetrical. The first of these polarities takes place in point-plane space, while the second has its origin in the five-dimensional manifold of linear complexes and throws its light into line space.

It is necessary to broaden the scope of this investigation in order to be able to seek solutions. It will be found that there are a number of solutions, but the question of whether *all* solutions have been found will be left open.

Let $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ be two polarities which are generated by two distinct non-degenerate surfaces of second degree in three-dimensional projective space. As polarities are involutory correlations, the combination of $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ to $\boldsymbol{P}_1\boldsymbol{P}_2$ or $\boldsymbol{P}_2\boldsymbol{P}_1$ yields a collineation, which need not necessarily be involutory, as can easily be demonstrated by counter examples. For simplicity of presentation, it will be assumed here that the collineation $\boldsymbol{P}_1\boldsymbol{P}_2$ (or $\boldsymbol{P}_2\boldsymbol{P}_1$) has four real invariant points and four real invariant planes, which are the vertices A, B, C, D and the faces α, β, γ, δ of a tetrahedron (Section 2.2.2, class (1a)). Polarity $\boldsymbol{P}_1$ assigns to the vertex A of this tetrahedron a plane α', which $\boldsymbol{P}_2$ must assign to the point A. Obviously, A and α' are polar relative to both $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. This applies in the same way to the remaining invariant points B, C, D, to which the planes β', γ', δ' are assigned. The plane point-field (BCD) is mapped into itself by the collineation. In each of the polarities $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ the point field (BCD) corresponds to a bundle of planes ($\beta'\gamma'\delta'$), whose centre point must be an invariant point of the collineation which does not lie in the point field. In accordance with the assumptions, it can only be the point A; Similarly, the plane (BCD) must be identical to α. (Otherwise $\boldsymbol{P}_1(A)$ or $\boldsymbol{P}_2(A)$ must lie in the field (BCD) and that would only be possible if $ABCD$ was a tangential tetrahedron or $\boldsymbol{P}_1$ or $\boldsymbol{P}_2$ was a null-polarity; both of these possibilities are excluded by our assumptions (see Sections 4.8 and 5.4.) Corresponding results apply for the remaining vertices and faces of the tetrahedron, which means that $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$ and $\delta' = \delta$. Each vertex of the tetrahedron is the pole of the opposite face of the tetrahedron, with respect to both $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. The invariant tetrahedron of a collineation, which is the product of two polarities is thus seen to be a *polar tetrahedron* (Section 4.8) of *both* polarities and thus a simultaneous invariant figure in three-dimensional projective space of both polarities.

According to the fundamental theorem (Theorem 2.2) of the projective geometry of three dimensional space, the invariant tetrahedron of a collineation is uniquely determined. Otherwise, the collineation would be the identity. It follows that the common polar tetrahedron of two polarities is also *uniquely* determined. Its more or less real *existence* is a consequence of the same theorem (see Section 2.2).

Argument by analogy allows us to apply what has been demonstrated in three-dimensional projective space to the five-dimensional manifold of linear complexes. Consider $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, two polarities in the five-dimensional manifold of linear complexes. Let one, say $\boldsymbol{P}_1$, be the polarity on the universal line quadric and let the other, $\boldsymbol{P}_2$, be a polarity relative to an arbitrarily selected, non-degenerate quadratic surface in the five-dimensional manifold of linear complexes. The exact structure of this surface and its associated polarity are not significant here. It is only required that the polarity assign to each linear complex a 4-manifold (or hyperplane) and conversely to each 4-manifold a complex, whereby relationships between the associated manifolds which are analogous to those in three-dimensional projective space apply.

The combination of two involutory correlations $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ to $\boldsymbol{P}_1\boldsymbol{P}_2$ or $\boldsymbol{P}_2\boldsymbol{P}_1$ yields a collineation in the five-dimensional manifold of linear complexes. It will be assumed in what follows that this correlation possesses six real invariant linear complexes and six real 4-manifolds of linear complexes (or hyperplanes). The invariant linear complexes will be designated K_1, K_2, K_3, K_4, K_5, K_6 and the invariant 4-manifolds will be designated M_1, M_2, M_3, M_4, M_5, M_6. These sets of six invariant complexes and invariant 4-manifolds are interrelated in a manner corresponding to the interrelationship between the points and planes of a polar tetrahedron. This totality is self-polar relative to *both* second degree structures. Considering the universal line quadric, this means: The complex K_1 is root complex of the 4-manifold M_1, which contains the complexes K_2, K_3, K_4, K_5, K_6 and is the intersection of the 4-manifolds M_2, M_3, M_4, M_5, M_6; the complexes K_2, K_3, K_4, K_5, K_6 are also reciprocal or «in involution» (that is, mutually null-invariant, see Chapter 14) to complex K_1. This applies similarly for all six complexes and 4-manifolds of the polar invariant sixfold figure. Accordingly, the six complexes K_1, K_2, K_3, K_4, K_5, K_6 are pairwise reciprocal. The same applies to the 4-manifolds M_1, M_2, M_3, M_4, M_5, M_6, if two 4-manifolds are regarded as being reciprocal precisely when their root complexes are reciprocal.

As invariant elements of a collineation in the five-dimensional manifold of linear complexes, the six complexes K_1, K_2, K_3, K_4, K_5, K_6 and the six 4-manifolds M_1, M_2,

M_3, M_4, M_5, M_6 are uniquely determined. It follows that they are also *uniquely* determined as the common invariant of two polarities (one of which is the polarity with respect to the universal line quadric) in the five-manifold of linear complexes.

It is evident that the *common invariants* of a polarity with respect to a second degree surface in three-dimensional projective space and polarity with respect to the universal line quadric must be line figures, as lines are the only elements which are common to three-dimensional projective space and the five-dimensional manifold of linear complexes.

It is reasonable to begin a search for common invariants of these two polarities by considering their already known self-polar figures: the polar tetrahedron and the polar sextuplet of complexes and 4-manifolds.

In what follows, only polarities with respect to reguli will be considered, because the properties of complexes with respect to surfaces of second degree were only proven for reguli (Chapters 8, 13,16). Extension of this study to include other polarities in three dimensional space requires systematic application of the theory of imaginary elements (see Kötter [1982], Kötter/Stoss [2008]).

Consider a polar tetrahedron of a regulus $\mathcal{R}$ of second degree (Section 4.8). Consider the hyperbolic congruences which are generated by each pair of opposite skew edges (it will be necessary to consider elliptical congruences if the polar tetrahedron is partially imaginary). Linear congruences carry pencils of linear complexes. A tetrahedron has three pairs of opposite edges, which are the special complexes of three pencils of complexes. Choose one complex from each of these three pencils, say K_1, K_2, K_3. The polarity with respect to $\mathcal{R}$ assigns uniquely to each of the complexes K_1, K_2, K_3 a further complex, say K_4, K_5, K_6. The pencils (K_1K_4), (K_2K_5) and (K_3K_6) are now the pencils which were referred to above. This follows immediately from Theorem 13.2 and the fact that the common polar lines of a regulus of second degree and a complex (or null-polarity) remain unchanged when the complex is exchanged with its polar complex with respect to the regulus.

The structure of six complexes which has been constructed is clearly self polar with respect to the regulus $\mathcal{R}$, as this applies to the pair of polar complexes within a single pencil. It will be shown below that these six complexes are also polar invariant with respect to the universal line quadric.

All lines which intercept the directrices of the pencil of complexes (K_1K_4) are common lines of the complexes K_1 and K_4. These directrices are, by construction, the opposite edges of the polar tetrahedron of $\mathcal{R}$. The directrices of the pencils (K_2K_5) and (K_3K_6) belong to these interceptors. It follows that the complexes K_1 and K_4 are reciprocal to the complexes K_2, K_5, K_3, K_6, which means that they are mutually null-invariant (Theorem 14.2). Correspondingly, the complexes K_2, K_5 are reciprocal to the complexes K_1, K_4, K_3, K_6 and K_3, K_6 to K_1, K_4, K_2, K_5. Thus the polarity with respect to the universal line quadric maps the system of six complexes K_1, K_2, K_3, K_4, K_5, K_6 into itself. So, for example, the pencil of complexes (K_1K_4) maps into a pencil of 4-manifolds, that is, maps into a 3-manifold $(K_2K_5K_3\mathrm{K}_6)$, as in three-dimensional projective space a polarity maps an edge of a polar tetrahedron into the pencil of planes which lie in the opposite edge. This establishes that the system of six complexes K_i (i = 1, 2, ... ,6) is polar invariant with respect to both the regulus $\mathcal{R}$ and the universal line quadric and thus a common invariant of both polarities.

A given regulus $\mathcal{R}$ has ∞^3 associated polar tetrahedra (without proof). It is possible to select a complex in each of the pencils in ∞^1 ways. The three further complexes are then determined uniquely by the polarity with respect to $\mathcal{R}$. It follows that there are altogether ∞^3 polar invariant complexes and thus ∞^6 such systems of six complexes for a given regulus $\mathcal{R}$. The uniqueness which was established for invariants of polarities of the same type has been lost.

A system of six special complexes, whose axes lie in the edges of a polar tetrahedron, is likewise a solution to the problem of common invariants of two polarities. These special complexes are trivially polar invariant with respect to $\mathcal{R}$ as they are with respect to the universal line quadric. Each special complex is reciprocal to itself and also to those remaining four special complexes whose axes intersect its axis.

The starting point for developing the system of six complexes K_i (i = 1, 2, ... ,6) which has been constructed was a polar tetrahedron, which is known to be polar invariant with respect to a regulus of second degree. The result was a system of complexes that consisted of six pairwise reciprocal linear complexes which is also polar invariant with respect to the universal line quadric. In the next section, the argument will be turned around: Given a special type of sextuple of complexes, the so called *fundamental complexes*, which is polar invariant with respect to the universal line quadric, what are the simultaneous invariants with a polarity with respect to a regulus of second degree? It will be seen that a system of fundamental complexes itself constitutes such an invariant.

18.2 Fundamental complexes, fundamental surfaces and fundamental tetrahedra

Definition *Fundamental complexes* are systems of six pairwise reciprocal linear complexes.

Fundamental complexes which will be designated F_1, F_2, F_3, F_4, F_5, F_6 are by definition polar invariant with respect to the universal line quadric.

Note: The common invariants , K_1, K_2, K_3, K_4, K_5, K_6, of two polarities in the five-dimensional manifold of linear complexes from section 8.1, one of which was the polarity with respect to the universal line quadric constitute a system of fundamental complexes.

Fundamental complexes can most easily be constructed from consideration of a 2-manifold of complexes. Let F_1 be a complex of a 2-manifold $\mathcal{B}$. This 2-manifold $\mathcal{B}$ contains a complete pencil (1-manifold) of complexes which are reciprocal to F_1, as a 4-manifold of complexes and a 2-manifold of complexes intersect in a pencil of complexes (Table 15.1). The complexes of this pencil can be ordered into involutory pairs of reciprocal complexes (Theorem 14.6). If such a pair of reciprocal complexes, say F_2 and F_3, is selected, then the three complexes F_1, F_2 and F_3 are pairwise reciprocal. Applying the same procedure in the 2-manifold of root complexes $\mathcal{B}'$ yields six pairwise reciprocal complexes F_1, F_2, F_3, F_4, F_5, F_6.

According to Theorem 13.1 a linear complex is self polar with respect to every regulus of second degree, in which one of the generating reguli consists of lines of the complex. Each of the complexes F_1, F_2, F_3, F_4, F_5, F_6 which have just been constructed contains one set of generators of the quadratic surface $\mathcal{R}$ determined by the polarity produced by the 2-manifold $\mathcal{B}$ (Sections 16.2 and 16.3). The fundamental complexes are thus common invariants of the polarity with respect to $\mathcal{R}$ and the polarity with respect to the universal line quadric. This means that they are a further solution to the problem of simultaneous invariants of two polarities. Again, the uniqueness which is characteristic of simultaneous invariants of polarities of the same type is not present.

Theorem 18.1 *A 2-manifold of complexes and thus a polarity with respect to a ruled surface has* ∞^6 *systems of fundamental complexes.*

Proof: A 2-manifold $\mathcal{B}$ contains ∞^2 complexes, each of which uniquely determines a reciprocal pencil of complexes within this 2-manifold. So ∞^1 pairs of reciprocal complexes can be constructed within these pencils (Theorem 14.6). Thus one arrives at ∞^3 triplets of pairwise reciprocal complexes in the 2-manifold $\mathcal{B}$. Applying the same procedure to the 2-manifold of root complexes $\mathcal{B}'$ leads to the asserted ∞^6 systems of fundamental complexes.

There are $\infty^{3 \cdot 5} / \infty^{3 \cdot 2} = \infty^9$ 2-manifolds and thus $\infty^6 \cdot \infty^9 = \infty^{15}$ systems of fundamental complexes in the five-dimensional manifold of linear complexes.

Any three of six fundamental complexes determine a 2-manifold and have a common regulus. There are therefore for any given system of six fundamental complexes twenty ways to construct a regulus. In this case each 2-manifold of root complexes generates the directrices of the reguli which consist of lines common to all complexes of the original 2-manifold (Theorem 16.6). Thus, the twenty reguli belong to ten ruled surfaces of second degree. These are the ten quadratic surfaces which are determined by the ten polarities. These ten polarities are generated by 2-manifolds each of which is determined by sets of three of the six fundamental complexes, because a 2-manifold $\mathcal{B}$ and the associated 2-manifold of root complexes $\mathcal{B}'$ generate the same polarity.

Definition The ten ruled surfaces of a system of six fundamental complexes are called *fundamental surfaces*.

By a combinatorial argument, any two of the six fundamental complexes determine a linear congruence, of which there are fifteen, with thirty (possibly imaginary) directrices. The two directrices of each such congruence are lines which are null polar with respect to each complex of the pencil determined by the congruence. They are also contained in all complexes which are reciprocal to this complex pencil. That is, they all belong to a 3-manifold of complexes. In particular, this 3-manifold contains the four remaining fundamental complexes. For example, if u and v are the two directrices of the congruence (F_1F_2) generated by the complexes F_1 and F_2, they are contained in all six of the congruences which are formed by the remaining four complexes F_3, F_4, F_5, and F_6, as each of these is reciprocal to both F_1 and F_2. Thus the lines u and v are intersected by the twelve directrices of these six congruences. (The directrices have always been referred to as being real. More detailed considerations show that all results which are presented here also apply when the directrices are imaginary. It is then necessary to interpret «intersection" and »connection" in the terms of the theory of imaginary elements.)

The two directrices of each of the congruences (F_1F_2), (F_3F_4) and (F_5F_6) are always intersected by the directrices of the other two congruences. It follows that the six directrices form a tetrahedron.

Definition Given three linear congruences which are determined by three disjunct pairs of linear complexes of a system of six fundamental complexes, the tetrahedron which is formed by the six directrices of these three congruences is called a *fundamental tetrahedron.*

A system of six fundamental complexes possesses fifteen fundamental tetrahedra. The proof is again purely combinatorial. The six fundamental complexes can be partitioned into three disjunct pairs in fifteen different ways. Because, one complex can be grouped with any one of the other five complexes. The remaining four complexes can form six different pairs, whereby one pair uniquely determines the other. Thus there are only three distinct ways for a given pair to choose the remaining two points and so it follows that there are $5 \cdot 3 = 15$ ways to form the required three disjunct pairs of complexes.

Each associated pair of directrices (i.e. *the* directrices of a chosen congruence) is intersected by $3 \cdot 4 = 12$ directrices (see above), which as just enumerated belong to three different pairs of two complexes. Thus the pair of directrices which is under consideration belongs to three fundamental tetrahedra.

An associated pair selected from the thirty directrices lies in four of the fundamental surfaces. So, for example, the pair of directrices of the congruence (F_1F_2) lies in each of the quadratic surfaces which are listed in Table 18.1. The triplets in this table specify 2-manifolds and the associated 2-manifolds of root complexes.

Table 18.1	Table 18.2
$((F_1F_2F_3), (F_4F_5F_6))$	$((F_1F_3F_4), (F_2F_5F_6))$
$((F_1F_2F_4), (F_3F_5F_6))$	$((F_1F_3F_5), (F_2F_4F_6))$
$((F_1F_2F_5), (F_3F_4F_6))$	$((F_1F_3F_6), (F_2F_4F_5))$
$((F_1F_2F_6), (F_3F_4F_5))$	$((F_1F_4F_5), (F_2F_3F_6))$
	$((F_1F_4F_6), (F_2F_3F_5))$
	$((F_1F_5F_6), (F_2F_3F_4))$

Theorem 18.2 *Given an associated pair of directrices among the thirty directrices of a system of six fundamental complexes.*

(1) *The two directrices are intersected by twelve of the remaining directrices.*

(2) *The two directrices lie in three fundamental tetrahedra and four fundamental surfaces.*

(3) *The two directrices are polar with respect to the remaining six fundamental surfaces*

Proof: (1) and (2) have just been proven. – (3) The assertion will be proven for the pair of directrices of the congruence (F_1F_2). The remaining cases are covered by cyclically advancing the subscripts of the complexes. (All ten fundamental surfaces are contained in Tables 18.1 and 18.2.) Inspection of Table 18.2 shows that F_1 and F_2 are not contained in the same 2-manifold, but that one lies in a 2-manifold $\mathcal{B}$ and the other lies in the associated 2-manifold of root complexes $\mathcal{B}'$. Thus, the directrices of (F_1F_2), u and v, do not belong to any of the common reguli of the complexes belonging to either of the 2-manifolds $\mathcal{B}$ or $\mathcal{B}'$, because only directrices of congruences of the 2-manifold $\mathcal{B}$ or of the associated 2-manifold of root complexes $\mathcal{B}'$ are, as axes of the special complexes in $\mathcal{B}$ or $\mathcal{B}'$, rulers of the fundamental surface. Both other pairs of complexes - (F_3F_4) and (F_5F_6) in the first line of Table 18.2 – yield two congruences $\mathcal{C}$ and $\mathcal{C}'$, whose directrices intersect the directrices of the congruence (F_1F_2) and thus form together with u and v a fundamental tetrahedron (Figure 18.1). One of the congruences $\mathcal{C}$, $\mathcal{C}'$ is in the 2-manifold $\mathcal{B}$, while the other is in the 2-manifold of root complexes $\mathcal{B}'$. Their directrices thus belong to the two reguli of a ruled surface. For example, consider the congruences $\mathcal{C} = (F_3F_4)$ and $\mathcal{C}' = (F_5F_6)$ for which the corresponding reguli are within the fundamental surfaces $((F_1F_3F_4), (F_2F_5F_6))$ and $((F_1F_5F_6), (F_2F_3F_4))$

respectively. Thus the lines u and v are polar and the corresponding fundamental tetrahedron is a tangential tetrahedron (Figure 18.1).

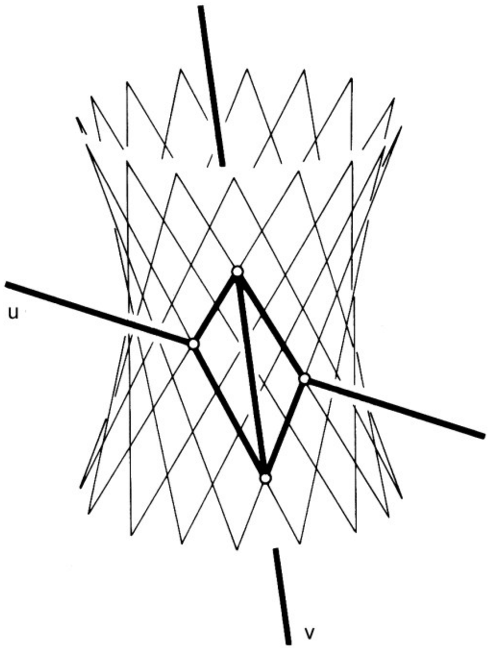

Figure 18.1

Theorem 18.3 *The ten fundamental surfaces of a system of six fundamental complexes form two groups with respect to a fundamental tetrahedron.*

(1) *The members of the first group of six surfaces each contain four of the six edges of the fundamental tetrahedron, which is consequently a tangential tetrahedron of these surfaces.*

(2) *The fundamental tetrahedron is self polar relative to the remaining six surfaces, and thus is a polar tetrahedron.*

As shown in Theorem 18.2, the two directrices of a congruence, say (F_1F_2), either (1) belong to the fundamental surface associated with triplets of complexes, to one of which F_1 and F_2 belong or (2) are a polar pair of lines with respect to a fundamental surface associated with triplets of complexes, neither of which contains both F_1 and F_2. This fact leads almost immediately to the desired result. – Simple combinatorics show that when three pairs are partitioned into two triplets, there are either no pairs in either triplet or one pair in each triplet. Of the ten possible cases, six yield one pair in each triplet and four yield no pair in either triplet. If, for example, the three pairs are (F_1F_2), (F_3F_4) and (F_5F_6), then all ten possible pairs of triplets are listed in Tables 18.1 and 18.2. Inspection of these tables identifies: (1) the six cases in which each triplet contains a pair: the four cases of Table 18.1 and the first and last cases of Table 18.2; (2) the four cases in which neither triplet contains a pair: the remaining four cases of Table 18.2. – When the pairs are interpreted as congruences and the triplets as fundamental surfaces, the six cases in (1) identify six tangential tetrahedra and the four cases in (2) identify four polar tetrahedra. The same argument applies to any choice of three congruences.

Theorem 18.4 *The fifteen fundamental tetrahedra of a system of six fundamental complexes form two groups relative to a fundamental surface. One group contains six polar tetrahedra, the other contains nine tangential tetrahedra.*

Proof: Consider, for example, the surface $((F_1F_2F_3), (F_4F_5F_6))$. Table 18.3 shows that there are six possible ways to form three pairs of complexes, so that one member of each pair belongs to a 2-manifold and the other belongs to the associated 2-manifold of root complexes. The six fundamental tetrahedra which are formed by the three pairs of directrices of the congruences associated with these three pairs of complexes are polar tetrahedra, as none of the pencils of complexes is contained within either the 2-manifold or the associated 2-manifold of root complexes. A combinatorial consideration shows that the remaining nine fundamental tetrahedra are tangential tetrahedra because four of their edges lie in the fundamental surface.

(F_1F_4)	(F_2F_5)	(F_3F_6)
(F_1F_4)	(F_2F_6)	(F_3F_5)
(F_1F_5)	(F_2F_4)	(F_3F_6)
(F_1F_5)	(F_2F_6)	(F_3F_4)
(F_1F_6)	(F_2F_4)	(F_3F_5)
(F_1F_6)	(F_2F_5)	(F_3F_4)

Table 18.3

The system of six pairwise reciprocal fundamental complexes was considered in Section 18.1 as a figure of the five-dimensional manifold of linear complexes. It appears there as a polar sixfold figure of complexes and 4-manifolds which is simultaneously invariant with respect to two polarities: with respect to the universal line quadric and with respect to another polarity in the five dimensional manifold of linear complexes. Although there has been no reference to 4-manifolds in this section, it is not thereby incomplete. The *complete* system of six reciprocal complexes and 4-manifolds has been considered. For the case of the five-dimensional manifold of linear complexes, it is sufficient to exchange the expressions «complex» and «4-manifold». In the much more important case of line-space, complexes, pencil of complexes and 2-manifolds of complexes cannot be distinguished from the 4-manifolds, 3-manifolds and 2-manifolds which are reciprocal to them (Section 17.4). The special complexes of the latter mentioned manifolds are the generators of the common line figures of the former -manifold of complexes.

18.3 Notes and references

A mathematically more complete treatment of projectivities in the five-dimensional manifold of linear complexes can be found, for example, in Stoss [1999].

Fundamental complexes play an important role in construction of co-ordinates in the five-dimensional manifold of linear complexes (Jessop [1903], Stoss [1995], [1999].

Klein discovered the system of six pairwise reciprocal mutually nullinvariant complexes when investigating projective invariants. Specifically, he was investigating linear transformations of coordinates for linear complexes which transform a quadratic function between such coordinates into normal form and leave the universal line quadric invariant. They are therefore also known as *Klein's Fundamental Complexes* (see Klein [1868], [1870a], [1870b], [1872]). This text adheres closely to Klein's ideas.

The 6-fold infinite possibilities in the choice of fundamental complexes with respect to a second degree surface obtain practical significance in the Mechanics of rigid bodies. It relates to the six degrees of freedom of any rigid body in Euclidean or non-Euclidean space, whose projective metric is determined by the second degree surface mentioned above (Adams [1977], [1966], Ball [1900]).

The theory and history of line geometry and mechanics are considered in Ziegler [1985], where further references are included.

For further consideration of imaginary theory within line geometry see Ziegler [1998], Juel [1934]. For imaginary theory relating to linear complexes refer to Kötter [1982], Kötter/Stoss [2008].

Further applications of the system of fundamental complexes can be found in Gschwind [1991], [2000], [2005], [2008].

Further references

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Acknowledgements

This chapter was translated by Robert Byrnes from the German original. He also made some suggestions for improving the text.

Publication of earlier parts of this series of papers in the journal «Mathematisch-Physikalische Korrepondenz»:

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3. Introduction to curves and surfaces in three-dimensional projective space: 2006, Nr. 225: 40–48.
4. Surfaces of the second degree in three-dimensional projective space: 2006, Nr. 226: 20–39.
5. Involutory collineations and polarities in three-dimensional projective space: 2006, Nr. 227: 23–39.
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11. Twisted cubics and cubic developables in three-dimensional projective space: 2008, Nr. 233: 32–48.
12. Collineations in three-dimensional projective space: tetrahedral quadratic complexes: 2008, Nr. 234: 45–56.

13a. Non-Euclidean, affine and Euclidean properties of linear families of lines in three-dimensional projective space. Part I: 2008, Nr. 235: 38–48.

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14. Reciprocal linear complexes in three-dimensional projective space: 2009, Nr. 237: 32–40.
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