Projectivities in Three-Dimensional Space

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This chapter gives a fairly complete treatment of important types of projectivities in three-dimensional projective space. The emphasis is on transformations, their classifications and characteristic properties, not on geometric figures. However, some important structures, particularly quadratic surfaces, linear congruences and complexes, are closely related to one or the other of these transformations. For surfaces of the second degree, these relationships will be presented in section 5.4.2; similar treatments for linear complexes and linear congruences have to be postponed until sections 8.5 and 9.6 respectively, that is, after they have been defined in terms of projective generations independent of these transformations.

After proving the three-dimensional analogue of the Fundamental Theorem of Projective Geometry (Theorem 1.1) in section 2.1, a complete classification of three-dimensional autocollineations is given in section 2.2. Some important classes of three-dimensional autocorrelations are treated in sections 5.2.2 and 5.3, and the properties of the most interesting classes of three-dimensional autocollineations are studied in sections 5.4 and 5.5.

2.1 Three-dimensional collineations and correlations

Definition *Three-dimensional projective space* is the set of all point, lines and planes that satisfy the axioms for primitive elements containing each other **(CO)**, the axioms of connection and intersection **(CI)** and the axioms of order and continuity **(OC)**.

It is useful within certain contexts to think of three-dimensional projectivities as occurring between two distinct spaces instead of being transformations of three-dimensional projective space into itself. In the last case, one has to deal with three-dimensional autocollineations and autocorrelations (see sections 2.2 and 5.2.2)

Definition A one-to-one correspondence between two three-dimensional projective spaces is called a *three-dimensional collineation* if to two dissimilar primitive elements a, B containing each other of one space correspond two dissimilar primitive elements a', B' containing each other of the other space, such that a, a' and B, B' are each pairs of different primitive elements.

Definition A one-to-one correspondence between two three-dimensional projective spaces is called a *three-dimensional correlation* if to two dissimilar primitive elements a, B containing each other of one space correspond two dissimilar primitive elements A', b' containing each other of the other space, such that a, A' and B, b' are each pairs of dissimilar primitive elements.

Definition A one-to-one correspondence between two three-dimensional projective spaces is called a *three-dimensional projectivity*, if it preserves the relation of primitive elements containing each other, or, equivalently, if one-dimensional primitive forms correspond to one-dimensional primitive forms such that they are projective.

The definition of linear dependence of points and planes from section 1.5 needs to be adjusted to three dimensions. Linear dependence of lines will be defined in section 7.3.

Definition Four points or four planes are called *linearly independent* if they do not lie in one two-dimensional primitive form, that is, if they do not have a plane or a point in common respectively. Five points or five planes are called *linearly independent*, if no four of them are linearly dependent.

Theorem 2.1A *A collineation between two three-dimensional projective spaces $\mathcal{P}$, $\mathcal{P}'$ is uniquely determined by two planes α, β in $\mathcal{P}$ that are collinear to two planes α', β' in $\mathcal{P}'$ respectively such that to every point lying on $\alpha\beta$ corresponds a point lying on $\alpha'\beta'$.*

Theorem 2.1a *A correlation between two three-dimensional projective spaces $\mathcal{P}$, $\mathcal{P}'$ is uniquely determined by two centric bundles A, B in $\mathcal{P}$ that are correlative to two planar fields α', β' in $\mathcal{P}'$ respectively such that to every plane passing through AB corresponds a point lying on $\alpha'\beta'$.*

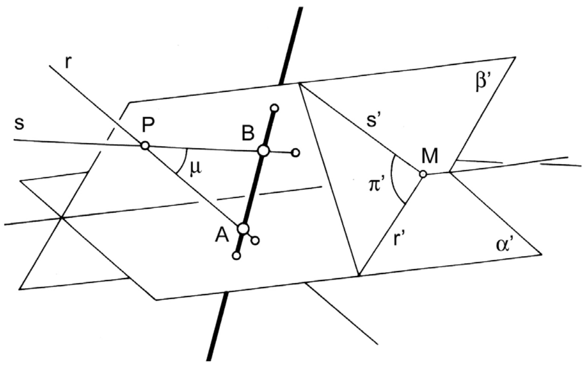

Figure 2.1

Proof (for the sake of variety the right side is proved; see Figure 2.1): Elements belonging to $\mathcal{P}'$ or $\mathcal{P}$ are denoted by primed or unprimed symbols respectively. The proof is carried out in two steps according to the following claims, where the various special cases are left as exercises for the reader. (i) To every point P corresponds a unique plane π' and to every line l passing through P corresponds a unique line l' lying in π'. – If P does not lie on AB, the lines $r = AP$ and $s = BP$ correspond with respect to the given correlations to some lines r' in α' and s' in β' which intersect in a point M' of $\alpha'\beta'$ that corresponds to the plane $\mu = APB$. The plane π' corresponding to P is now defined as the plane determined by $r's'$. Any line l not coinciding with AB determines with A and B two planes Al and Bl respectively that correspond to two points on α' and β' respectively which determine the line l'. If l passes through P, the planes Al and Bl belong respectively to the pencils of planes through AP and BP, hence the two points corresponding to these planes lie on $r' = \alpha'\pi'$ and $s' = \beta'\pi'$ respectively, hence l' lies on π'. (ii) To every plane ε which passes through a line l or through a point P, there corresponds a point E' which lies on the corresponding line l' or on the corresponding plane π', that is, ε and E' are correlative. – One can define a perspectivity between the bundles A, B by projecting ε from A, B such that the planes passing through AB correspond to themselves. Since then the bundle A is perspective to the bundle B and α' is projective to A as well as β' projective to B, α' must be projective to β' such that the points of $\alpha'\beta'$ correspond to themselves. Hence the planes α' and β' are perspective with respect to a bundle with center E' (Theorem 1.19A). The point E' is defined as the point corresponding to ε and is the center of a bundle that is correlative to the field ε. – In conclusion, corresponding points and planes are correlative, hence the theorem is proved.

Now the *Fundamental Theorem of Three-dimensional Projective Geometry* can be proved.

Theorem 2.2A *A collineation between two three-dimensional projective spaces is uniquely determined if to the elements of a set of five linearly independent points or planes in one space correspond uniquely the elements of a set of five linear independent points or planes respectively in the other space.*

Theorem 2.2a *A correlation between two three-dimensional projective spaces is uniquely determined if to the elements of a set of five linearly independent points or planes in one space correspond uniquely the elements of a set of five linear independent planes or points respectively in the other space.*

Proof (for correlations): Take A, B, C, D, E as five linear independent points of one space and α', β', γ', δ', ε' respectively as the corresponding five linearly independent planes of the other space. Theorem 2.2a can be reduced to Theorem 2.1a by demonstrating that, for example, A and B can be made correlative to α' and β' respectively such that the planes of AB correspond to the points of $\alpha'\beta'$. The correlation between A and α' (B and β') is uniquely determined by ordering the lines AB, AC, AD, AE (BA, BC, BD, BE) to the lines $\alpha'\beta'$, $\alpha'\gamma'$, $\alpha'\delta'$, $\alpha'\varepsilon'$ ($\beta'\alpha'$, $\beta'\gamma'$, $\beta'\delta'$, $\beta'\varepsilon'$) respectively. From this follows that the planes ABC, ABD, ABE belong to both bundles with centers A, B and correspond to the three points $\alpha'\beta'\gamma'$, $\alpha'\beta'\delta'$, $\alpha'\beta'\varepsilon'$ that are common to both fields α', β'. Hence, by Theorem 1.1, every plane of the pencil AB corresponds to a point of the range $\alpha'\beta'$.

In fact, Theorem 2.2 is equivalent to Theorem 2.1, that is, Theorem 2.1 can also be deduced from Theorem 2.2. This is a good exercise and is left to the reader. Note in passing that, since the product of two correlations is a collineation, Theorem 2.2A is an immediate consequence of Theorem 2.2a.

2.2 Classification of three-dimensional autocollineations

2.2.1 *Invariant elements of three-dimensional autocollineations*

A three-dimensional autocollineation is a collineation of a three-dimensional projective space into itself. Later in this chapter, the prefix «auto» will be dropped if there is no danger of confusion.

As in two dimensions, the following Theorem 2.3 is equivalent to the uniqueness property of Theorem 2.1. The proof is perfectly analoguous to the proof of the Theorem 1.15 given in section 1.6 for the two-dimensional case and hence needs not to be repeated here.

Theorem 2.3A *If a three-dimensional autocollineation leaves five linearly independent points invariant, it is the identity.*

Theorem 2.3a *If a three-dimensional autocollineation leaves five linearly independent planes invariant, it is the identity.*

One can conclude from this that an autocollineation that is not the identity has at most four linearly independent proper (i.e. real) invariant points or planes. But there is no lower limit $k > 0$ for the number $n \geq 0$ of proper invariant points or planes as in the two-dimensional case (Theorem 1.16). As will be seen in the next section, there are collineations that have no proper invariant points or planes at all. However, even in this case, there are always at least two proper invariant lines (that may coincide). But as this is rather difficult to prove and needs powerful tools of line geometry, it is postponed to section 12.4 (Theorem 12.13).

Before one can delve into the classification of autocollineations, one needs some information as to how the various invariant elements are related to each other. Once again, elements which are not explicitly called improper are considered to be proper.

Theorem 2.4A *If a non-identical three-dimensional autocollineation has an invariant point then it has an invariant line and an invariant plane that are contained in it.*

Theorem 2.4a *If a non-identical three-dimensional autocollineation has an invariant plane then it has an invariant line and an invariant point that are contained in it.*

Proof: Immediate consequence of the Theorems 1.16 and 1.17.

Theorem 2.5A *If a non-identical three-dimensional autocollineation leaves the points of a range invariant then it leaves the planes of a pencil invariant.*

Theorem 2.5a *If a non-identical three-dimensional autocollineation leaves the planes of a pencil invariant then it leaves the points of a range invariant.*

Proof (left side): Assume that u is the base-line of a range of invariant points. Then the pencil of planes with base u is transformed into itself. If this collineation is not the identity (which would prove the theorem) then there are at least two pairs α, α' and β, β' of corresponding planes passing through u. Since each pair contains u, α and α' as well as β and β' are perspective and hence are sections of bundles with centers A and B respectively (Theorem 1.19A). Every plane common to both bundles A, B corresponds to itself since it contains a line each in α and β and also their corresponding lines in α', β'. (Hence, the center of perspectivity of every pair of corresponding planes passing through u must lie on AB.)

Let u be a range of invariant points and v the base of the corresponding pencil of invariant planes. The lines u and v are, in general, skew, but they may also intersect each other or coincide (see next section). If v is also a range of invariant points then u is also the base of a pencil of invariant planes. In this case, u and v have to be either skew or coincide, otherwise, i.e. if they intersect without coinciding, they determine a field of invariant points and hence also a bundle of invariant planes.

Theorem 2.6A *If a non-identical three-dimensional autocollineation leaves the points and lines of a field invariant, then it also leaves the planes and lines of a bundle invariant.*

Theorem 2.6a *If a non-identical three-dimensional autocollineation leaves the planes and lines of a bundle invariant, then it also leaves the points and lines of a field invariant.*

Proof (left side): Let ε be the plane of invariant elements. Two corresponding planes α and α' will intersect in a line $\alpha\alpha'$ which lies in ε (Figure 2.2). Hence α and α' are perspective and thus sections of a bundle with center Z (Theorem 1.19A). But every line or every plane through Z intersects ε in an invariant point or line respectively and likewise intersects the planes α and α' in corresponding points or corresponding lines. Therefore, the bundle Z consists entirely of invariant elements.

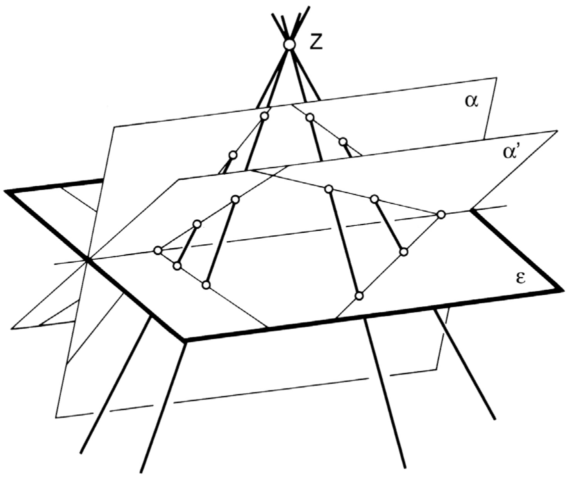

Figure 2.2

The Theorems 2.4, 2.5 and 2.6 imply that there are as many points or planes in a set of invariant elements of a three-dimensional autocollineation as there are invariant planes or points respectively in this set (Theorem 2.7).

Theorem 2.7 *The points and planes of a set of invariant elements of a three-dimensional autocollineation can be related by a one-to-one correspondence.*

In formulating the theorems of this section one has already touched upon the most important cases of three-dimensional autocollineations. The following definition describes the ones that were given specific names.

Definition A three-dimensional autocollineation that has exactly one range of invariant points (and hence exactly one pencil of invariant planes) is called *uniaxial*.

Definition A three-dimensional autocollineation that leaves exactly the points and lines of a field (and hence the lines and planes of a bundle) invariant is called *perspective*.

Definition A non-perspective three-dimensional autocollineation in which every line that joins two corresponding points or is the intersection of two corresponding planes is invariant is called *biaxial* or *skew*.

Uniaxial collineations will not be treated in any detail in this series of papers. Three-dimensional perspective collineations will be treated in section 2.3 and some special classes of biaxial collineations will appear in section 5.1.3. Note in passing: Three-dimensional non-perspective collineations that have two skew ranges of invariant points are biaxial, hence the name; however, the converse is not true, that is, the axes of a biaxial collineation need not be proper, or real, in any case.

2.2.2 *Classification of three-dimensional autocollineations*

The theorems of the foregoing section provide the background one needs in order to classify three-dimensional autocollineations completely. Particularly, Theorem 2.5 will be applied frequently without mentioning this fact every time explicitly. In addition, one often has to use the elementary facts that two invariant points determine an invariant line, an invariant line and an invariant point not contained in it determine an invariant plane, etc.

The following classification of autocollineations is based on properties of proper invariant elements. The classification process is started with the following distinction that divides all collineations in three classes:

(A) There exists at least one proper invariant point A_1 and one proper invariant plane ε_1 not passing through A_1.

(B) There exists at least one proper invariant point and one proper invariant plane; each invariant point lies in each invariant plane.

(C) There are no proper invariant points and no proper invariant planes.

To begin with, assume that there is at least one invariant point and hence, by duality, at least one invariant plane. However, according to Theorem 2.3, there can be no more than four linearly independent invariant points or planes. In the subsequent derivation, linearly independent invariant points and planes respectively are throughout denoted by A_1, A_2, A_3, A_4 and ε_1, ε_2, ε_3, ε_4 such that A_i and ε_i, $i = 1, 2, 3, 4$, are points and planes which do *not* contain each other. Every class of collineations is denoted by a latin numeral in parentheses that refers at the same time to a diagram in Figure 2.3 that represents the invariant elements of this particular class in a, hopefully, immediate comprehensible way. In addition, for every individual class the class of autocollineations of its invariant planes is indicated by writing, for example, ε_3 **(II)** or $\varepsilon_1\varepsilon_3$ **(II)** if ε_3 or all planes of pencil $\varepsilon_1\varepsilon_3$ are transformed by two-dimensional autocollineations of class **(II)** (see section 1.7). Since A_i is the only invariant point which does *not* lie on ε_i, it follows that the class of autocollineations of the bundle with center A_i is the same as the class of autocollineations of the invariant plane ε_i, $i = 1, 2, 3, 4$.

(A) There exists at least one proper invariant point A_1 and one proper invariant plane ε_1 that does not pass through A_1. Hence the collineation in plane ε_1 must belong to any of the classes **(Ia)**, **(Ib)**, **(II)**, **(III)**, **(IV)**, **(V)**, **(VI)**. This yields the following classes.

(A.Ia) Let ε_1 be of class **(Ia)** and let A_2, A_3, A_4 and ε_2, ε_3, ε_4 be the invariant points or invariant planes that contain ε_1 or A_1 respectively. If there are no additional invariant points (or planes), this yields the following class **(1a)**.

Class* (1a): *General collineation
Invariant points:
A_1, A_2, A_3, A_4
Invariant planes:
ε_1 **(Ia)**, ε_2 **(Ia)**, ε_3 **(Ia)**, ε_4 **(Ia)**
Invariant lines:
all six edges of the tetrahedron A_1, A_2, A_3, A_4

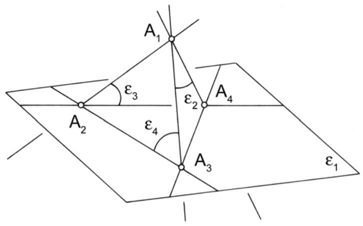

Figure 2.3: (1a)

Additional invariant points can only lie on the lines A_1A_2, A_1A_3, and A_1A_4. Assume that exactly one of these lines, say A_1A_2, is a range of invariant points, then one has class **(2a)**.

Class* (2a): *Uniaxial collineation
Invariant points:
A_3, A_4 and range A_1A_2
Invariant planes:
ε_3 **(III)**, ε_4 **(III)**, and pencil $\varepsilon_1\varepsilon_2$ **(Ia)**
Invariant lines:
A_1A_2, A_3A_4 and pencils (A_3, ε_4), (A_4, ε_3)

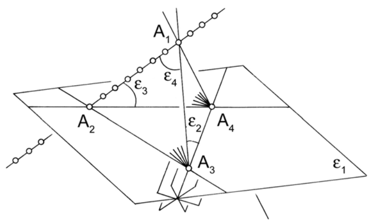

Figure 2.3: (2a)

If more than one invariant line outside ε_1 is pointwise invariant, then ε_1 could not be of class **(Ia)**, in contradiction the assumption above.

(A.Ib) Let ε_1 be of class **(Ib)** and take A_2 and $\varepsilon_1\varepsilon_2$ as the invariant elements in ε_1. If there are no invariant points other than A_1, A_2, one has the class **(1b)**.

***Class* (1b)**
Invariant points:
A_1, A_2
Invariant planes:
ε_1 **(Ib)**, ε_2 **(Ib)**
Invariant lines:
A_1A_2, $\varepsilon_1\varepsilon_2$

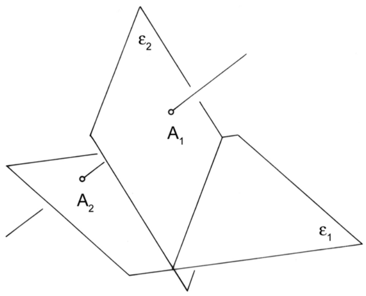

Figure 2.3: (1b)

Without disturbing the class of ε_1, additional invariant points can only sit on A_1A_2, hence one has class **(2b)**.

Class* (2b): *Uniaxial collineation

Invariant points:
range A_1A_2
Invariant planes:
pencil $\varepsilon_1\varepsilon_2$ **(Ib)**
Invariant lines:
A_1A_2, $\varepsilon_1\varepsilon_2$

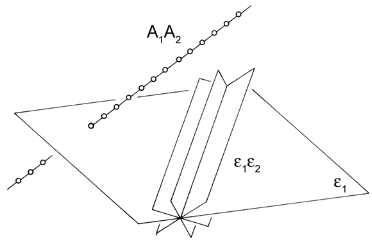

Figure 2.3: (2b)

(A.II) Let ε_1 be of class **(II)** and take A_2, A_4 as the invariant points and A_2A_4, $\varepsilon_1\varepsilon_2$ as the invariant lines in ε_1. If there are no invariant points other than A_1, A_2, A_4, one has class **(3a)**.

***Class* (3a)**

Invariant points:
A_1, A_2, A_4
Invariant planes:
ε_1 **(II)**, ε_2 **(II)**, ε_3 **(Ia)**
Invariant lines:
A_1A_2, A_1A_4, A_2A_4, $\varepsilon_1\varepsilon_2$

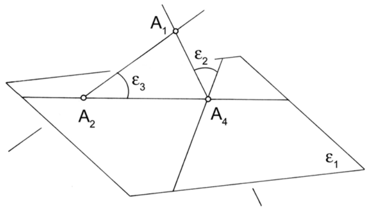

Figure 2.3: (3a)

Without disturbing the class of ε_1, additional invariant points can only lie on either A_1A_2 or A_1A_4. This yields the two classes **(4)** and **(5)**.

Class* (4): *Uniaxial collineation

Invariant points:
A_4, range A_1A_2
Invariant planes:
ε_3 **(III)**, pencil $\varepsilon_1\varepsilon_2$ **(II)**
Invariant lines:
A_1A_2, $\varepsilon_1\varepsilon_2$, pencil ($A_4$, ε_3)

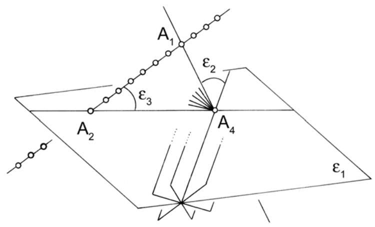

Figure 2.3: (4)

Class* (5): *Uniaxial collineation
Invariant points:
A_2, range A_1A_4
Invariant planes:
ε_1 **(II)**, ε_2 **(V)**, pencil $\varepsilon_1\varepsilon_3$ **(II)** with ε_3 **(III)**
Invariant lines:
pencils (A_2, ε_3), (A_4, ε_2)

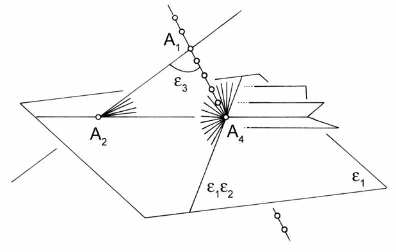

Figure 2.3: (5)

(A.III) Let ε_1 be of class **(III)** and take A_4 as the center and A_2A_3 as the axis of the homology in ε_1. If A_1 is the only invariant point outside ε_1, then one has again class **(2a)**. Without disturbing the class of ε_1, additional invariant points can either be in the plane $\varepsilon_4 = A_1A_2A_3$, or in the line A_1A_4. In the first case, plane ε_4 consists only of invariant elements, hence one has the following two classes **(6)** and **(7a)**.

Class* (6): *Perspective collineation (homology)
Invariant points:
A_4, field $\varepsilon_4 = A_1A_2A_3$
Invariant planes:
ε_4 **(VI)**, bundle $A_4 = \varepsilon_1\varepsilon_2\varepsilon_3$ **(III)**
Invariant lines:
field ε_4, bundle A_4

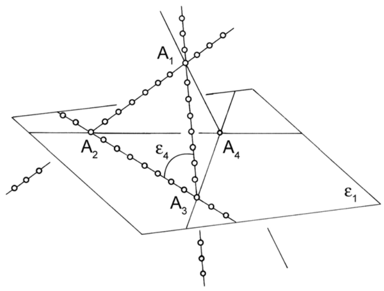

Figure 2.3: (6)

Class* (7a): *Hyperbolic biaxial collineation
Invariant points:
ranges A_1A_4, A_2A_3
Invariant planes:
pencils $\varepsilon_1\varepsilon_4$ **(III)**, pencils $\varepsilon_2\varepsilon_3$ **(III)**
Invariant lines:
A_1A_4, A_2A_3, and all lines that meet both A_1A_4 and A_2A_3

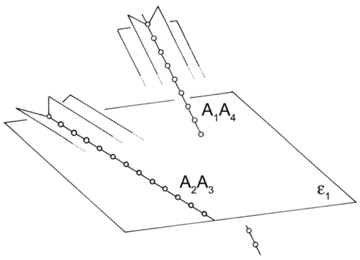

Figure 2.3: (7a)

(A.IV) Let ε_1 be of class **(IV)** and take A_4 as the invariant point and A_3A_4 as the invariant line. Either A_1 is the only invariant point outside ε_1, or an additional invariant point lies on A_1A_4 (otherwise one would need to alter the class of ε_1), hence (see Theorem 2.5) one arrives at the classes **(8)** and **(9)**.

***Class* (8)**
Invariant points:
A_1, A_4
Invariant planes:
ε_1 **(IV)**, ε_2 **(II)**
Invariant lines:
A_1A_4, $\varepsilon_1\varepsilon_2$

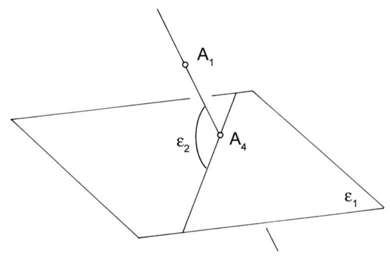

Figure 2.3: (8)

Class* (9): *Uniaxial collineation
Invariant points:
range A_1A_4
Invariant planes:
pencil $\varepsilon_1\varepsilon_2$ **(IV)** with ε_2 **(V)**
Invariant lines:
pencil (A_4, ε_2)

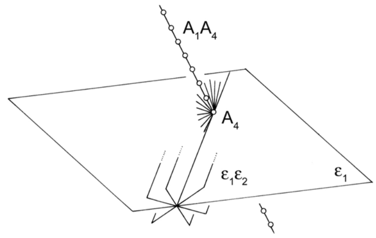

Figure 2.3: (9)

(A.V) Let ε_1 be of class **(V)** and take A_4 as the center and A_2A_4 as the axis of the elation in ε_1. If there are no other invariant points outside ε_1 besides A_1, one has again class **(5)**. In order not to disturb the class of ε_1, any additional invariant point must lie on $\varepsilon_3 = A_1A_2A_4$, hence making this plane consist only of invariant elements: class **(10)**.

Class* (10): *Perspective collineation (elation)
Invariant points:
field $\varepsilon_3 = A_1A_2A_4$
Invariant planes:
bundle $A_4 = \varepsilon_1\varepsilon_2\varepsilon_3$ **(V)** with ε_3 **(VI)**
Invariant lines:
field ε_3 and bundle A_4

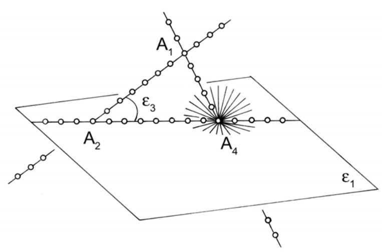

Figure 2.3: (10)

(A.VI) Let ε_1 be of class **(VI)**. If A_1 is the only invariant point outside ε_1, then one has again class **(6)**. Any additional invariant point outside ε_1 yields the identity, class **(11)**.

Class* (11): *Identity
Invariant points:
three-dimensional space of ∞^3 points
Invariant planes:
three-dimensional space of ∞^3 planes
Invariant lines:
∞^4 lines within three-dimensional space of points and planes

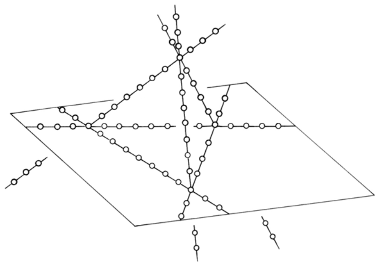

Figure 2.3: (11)

This exhausts all cases of class **(A)**.

(B) Three-dimensional autocollineations with at least one proper invariant point and one proper invariant plane such that each invariant point of this autocollineation lies in each invariant plane. Class **(B)** can be subdivided into the following subclasses:

(Ba) There are at least two invariant planes, say ε_1 and ε_3.
(Bb) There is exactly one invariant plane, say ε_1.

(Ba) All invariant points must lie on ε_1 *and* ε_3, that is, the possible autocollineations of the planes ε_1 and ε_3 can only belong to the classes **(II)**, **(IV)**, **(V)** and must, as it turns out, be the same for both invariant planes. Hence one has to consider the following cases:

(Ba.II) Let ε_1 be of class **(II)** and let A_2A_4 and A_3A_4 be the invariant lines and A_2, A_4 the invariant points of ε_1. This induces in ε_3 a collineation of class **(II)** such that A_2A_4 is an invariant line and the other invariant line passes through A_2 (it cannot pass through A_4, since this would imply an invariant plane not passing through A_2, in contradiction with **(B)**). Hence one has class **(12a)**.

***Class* (12a)**
Invariant points:
A_2, A_4
Invariant planes:
ε_1 **(II)**, ε_3 **(II)**
Invariant lines:
A_2A_4, A_2A_1, A_4A_3

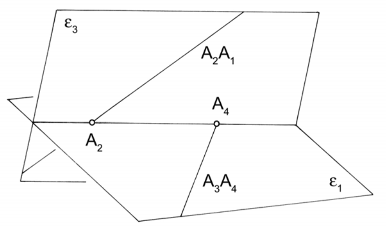

Figure 2.3: (12a)

(Ba.IV) Let ε_1 be of class **(IV)** and let A_2 be the invariant point and $\varepsilon_1\varepsilon_3$ the invariant line of ε_1. Hence the plane ε_3 must be of the same class than ε_1 and hence has the same invariant point A_2. But then the bundle A_2 contains exactly two distinct invariant planes which implies that there are exactly two distinct invariant points (Theorem 2.4). Since the invariant line $\varepsilon_1\varepsilon_3$ contains by assumption only one invariant point, this additional point must lie outside $\varepsilon_1\varepsilon_3$. Hence it is not the case that both invariant points can be contained in both invariant planes – in contradiction to **(B)**. Therefore there is no class that has the property **(Ba.IV)**.

(Ba.V) Let all points of $\varepsilon_1\varepsilon_3$ be invariant and hence both ε_1 and ε_3 are of class **(V)**. Let A_4 be the center for the elation in ε_1. Then the center for ε_3, say A_2, cannot coincide with $A_{4,}$ because if this were the case, all planes through A_4 would be invariant planes whereas only the planes of the pencil $\varepsilon_1\varepsilon_3$ contain all invariant points – in contradiction to **(B)**. The pencil of invariant planes (Theorem 2.5) must coincide with the pencil of $\varepsilon_1\varepsilon_3$ in order to comply with **(B)**. All these invariant planes are of class **(V)** such that their centers are distinct from each other (for the same reason as for the centers of ε_1 and ε_3). Hence one has class **(13)**.

***Class* (13)**
Invariant points:
range A_2A_4
Invariant planes:
pencil $\varepsilon_1\varepsilon_3$ **(V)**
Invariant lines:
every point of A_2A_4 is center of a pencil of invariant lines lying in its corresponding invariant plane (forming a parabolic linear congruence)

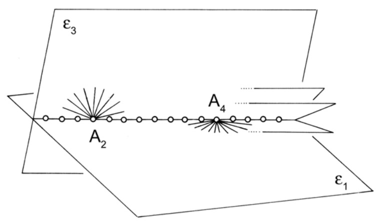

Figure 2.3: (13)

(Bb) There is exactly one invariant plane, say ε_1. It can be inferred from Theorem 2.4 that if there is exactly one invariant plane then there is exactly one invariant point. Therefore, the invariant plane ε_1 can only be of class **(Ib)** or **(IV)**.

(Bb.Ib) Let ε_1 be of class **(Ib)** and let A_2 be the invariant point and A_3A_4 the invariant line in ε_1. The only additional invariant line (Theorem 2.7) must pass through A_2 without lying in ε_1. This yields class **(3b)**.

***Class* (3b)**
Invariant point:
A_2
Invariant plane:
ε_1 **(Ib)**
Invariant lines:
A_1A_2, A_3A_4

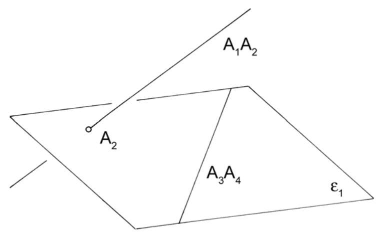

Figure 2.3: (3b)

(Bb.IV) Let ε_1 be of class **(IV)** and let A_4 be the invariant point in ε_1 and A_3A_4 the invariant line in ε_1. There can be no additional invariant line without producing additional invariant points or planes, hence one has finally class **(14)**.

***Class* (14)**
Invariant point:
A_4
Invariant plane:
ε_1 **(IV)**
Invariant line:
A_3A_4

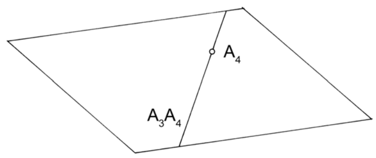

Figure 2.3: (14)

This classification exhausts all cases of three-dimensional autocollineations that have at least one proper (real) invariant point and hence one proper (real) invariant plane. Since all cases that involve two-dimensional autocollineations of class **(Ib)** were included systematically, this classification encompasses all cases of three-dimensional autocollineations the invariant figures of which are *partially* improper (imaginary or complex). Particularly, these are the classes **(1b)**, **(2b)**, and **(3b)**. They have the same numbers of invariant elements as the classes **(1a)**, **(2a)**, and **(3a)** respectively if one ignores the difference between proper and improper elements.

(C) In order to extend the classification to the cases in which *all* invariant points or planes are improper, one has to drop the assumption that there is at least one proper invariant point or plane as in the classes **(A)** or **(B)**. Therefore, one needs a new approach that does not rely on the considerations above which were carried out essentially by studying the properties of proper invariant *points* and *planes*. Proper invariant points or planes need not exist. But what about invariant lines? Do they always exist? In fact, they do; this can be seen by inspection in the cases where one has proper invariant points or planes also (see above). If there are no proper (real) invariant points or planes, the situation is much more complicated: the proof of the existence of invariant lines of a three-dimensional collineation without proper (real) invariant points or planes involves advanced methods of line geometry (see section 12.4). The result, however, is quite simple (Theorem 12.13): *If a three-dimensional autocollineation has no proper (real) invariant points or planes it has either two proper invariant lines, each of which contain a pair of improper (conjugate imaginary) invariant points (i.e. an invariant elliptic point involution), or one proper invariant line that contains one pair of improper (conjugate imaginary) invariant points*. Hence one has immediately the two additional classes of three-dimensional autocollineations, class **(1c)** and **(12b)**.

***Class* (1c)**

Proper invariant points:
none

Proper invariant planes:
none

Invariant lines:
A_1A_2, A_3A_4

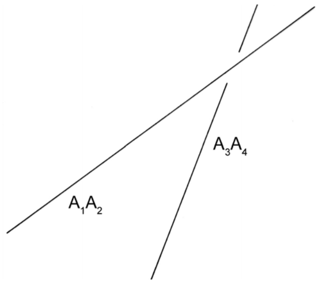

Figure 2.3: (1c)

***Class* (12b)**

Proper invariant points:
none

Proper invariant planes:
none

Invariant line:
A_2A_4

Figure 2.3: (12b)

Once again, the classes **(1c)** and **(12b)** are equivalent to the classes **(1a)** and **(12a)** respectively, since they both have the same number of invariant points and planes if improper invariant elements are treated in the same way as proper ones.

It will be shown in section 5.1.3 that there is a class of collineations in three-dimensional projective space that has no proper (real) invariant points or planes yet ∞^2 invariant lines. This class is closely related to class **(7a)**, hence one has class **(7b)**.

Class* (7b): *Elliptic biaxial collineation

Proper invariant points:
none

Proper invariant planes:
none

Invariant lines:
form an elliptic linear congruence

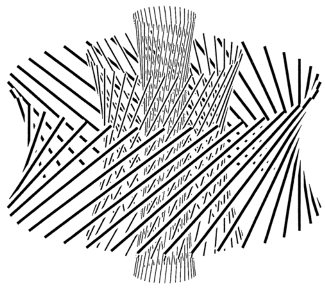

Figure 2.3: (7b)

Theorem 2.8 *There are 14 classes of three-dimensional autocollineations (including the identity) if one ignores if the invariant elements are proper (real) or not; otherwise, there are 20 classes.*

The astute reader may have noted that there are no existence proofs for all the classes mentioned above. This can be remedied either by giving examples (see Baldus [1928] or Grassmann [1896], pp. 438–464) or explicit constructions. The latter approach can be executed by applying Theorem 2.2 appropriately. For example, consider a collineation with four linearly independent invariant points, that is, four coinciding pairs of corresponding points. Then the collineation is perspective, biaxial, uniaxial, or general if the line joining a fifth pair of corresponding points passes through an invariant point, or intersects two invariant lines, or intersects exactly one invariant line, or does not contain any invariant element respectively (see Figure 2.4). There will be no discussion of the existence of the remaining classes here. However, the various kinds of collineations treated in chapter 6 provide explicit constructions for many additional classes, encompassing the most important ones.

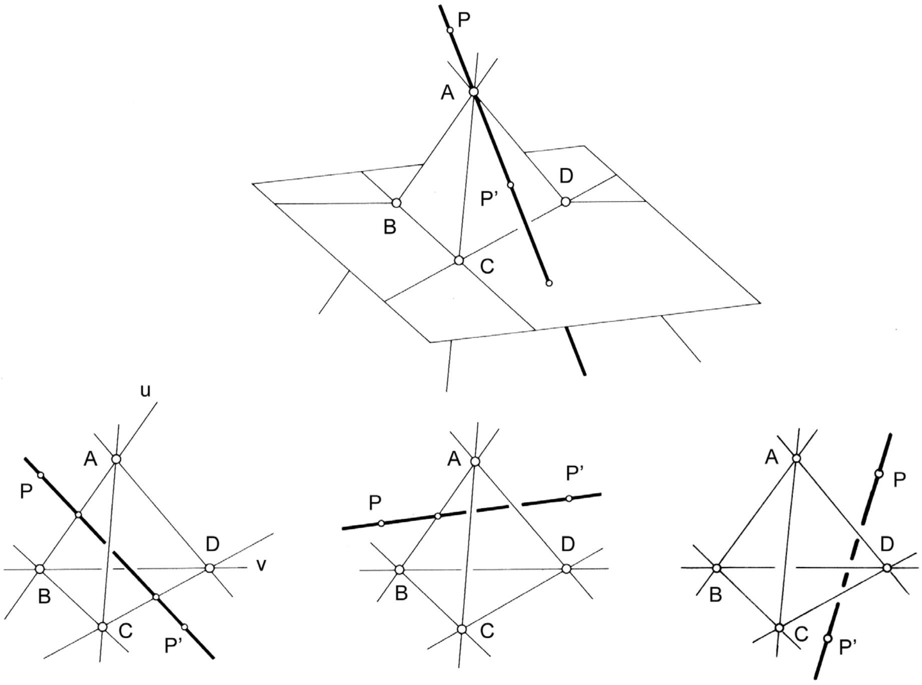

Figure 2.4

2.3 Three-dimensional perspective collineations

A three-dimensional collineation is called *perspective*, or a *perspectivity* if it has the points of a plane as invariant elements (see section 2.2.1). According to Theorem 2.6 the set of invariant elements of a perspectivity consists of all the elements of a bundle and a field.

Definition The center-point of the invariant planes of a three-dimensional perspectivity is called the *center of the perspectivity* and the base-plane of the field of invariant points the *axial plane of the perspectivity*. If the center lies on the axial plane, the perspectivity is called an *elation*, otherwise a *homology*.

Therefore, any two corresponding points P, P' of a perspectivity lie in an invariant line through the center Z and any two corresponding lines g, g' lie in a plane through Z and intersect in a point of the axial plane ζ; two corresponding planes meet in a line of the axial plane ζ (see Figure 2.5 for the case of a homology). The existence of perspectivities is confirmed by the following Theorem 2.9.

Theorem 2.9 *A perspectivity is uniquely determined by its center, its axial plane and a pair of distinct corresponding points or planes.*

Proof: Let Z be the center and ζ the axial plane of the perspectivity, and let A, A' and α, α' be corresponding points or planes respectively. If B, C, D are any linearly independent points in ζ and β, γ, δ any linearly independent planes through Z, then the projectivity between $ZABCD$ and $ZA'BCD$ or between $\zeta\alpha\beta\gamma\delta$ and $\zeta\alpha'\beta\gamma\delta$ uniquely determines a collineation (Theorem 2.2). This collineation is perspective, since the intersection point E of AA' with ζ is a fourth invariant point in ζ (see Figure 2.6), or, respectively, the plane $\alpha\alpha'Z$ is a fourth invariant plane through Z.

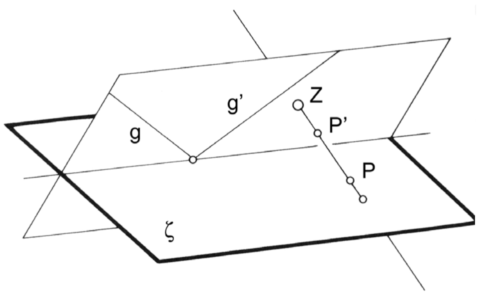

Figure 2.5

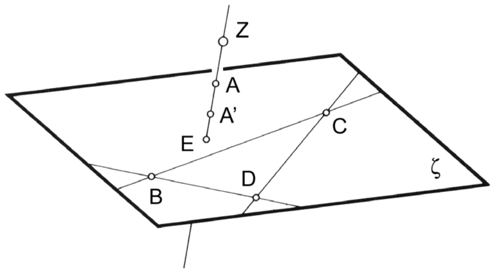

Figure 2.6

2.4 Notes and references

For a comprehensive treatment of projectivities in three-dimensional space, see Reye [1907] or Sturm [1909] or, from an algebraic point of view, Semple/Kneebone [1952]. The classification of three-dimensional autocollineations given here is mainly due to Baldus [1928], but see also Grassmann [1896], Newson [1897] [1900]. For an elementary introduction, see Edwards [1985]. For an algebraic treatment on the basis of path curves, see Boer [2004].

With minor adaptions, Figure 2.3: (7b) is taken from Pottmann/Wallner [2001], Figure 3.7, p. 176.

Publication of earlier parts of this series of papers in the journal «Mathematisch-Physikalische Korrepondenz»:

Introduction, references, and index: 2005, Nr. 222, p. 31–48.

1. Projectivities between primitive forms of one and two dimensions: 2005, Nr. 223, p. 35–48.

Reprint from:

Mathematisch-Physikalische Korrepondenz 2005, 224: 35–48.