Selected Topics in Three-Dimensional Synthetic Projective Geometry
Chapter 1

Projectivities Between Primitive Forms of One and Two Dimensions

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The concept of projectivity will be explained in section 1.1, where also the Fundamental Theorem of Projective Geometry is stated (Theorem 1.1). The latter is, in fact, the basis in which all higher projective geometry is rooted. Some of its consequences will be presented in sections 1.2 and 1.3. Projectivities of one-dimensional primitive forms into themselves are of special interest, and among these the one-dimensional involutions, discussed in section 1.4, are particularly useful.

In the rest of this chapter a brief account of two-dimensional projective geometry is given. There is no need to be exhaustive or even detailed since there are many good textbooks and monographs available on this subject. The emphasis lies more on two-dimensional projectivities as they occur within three-dimensional projective space, in contrast to the usual treatment of two-dimensional projective geometry as pure plane geometry.

1.1 The Fundamental Theorem of Projective Geometry

The reader is assumed to be familiar with the concept of harmonic sets of four elements on one-dimensional primitive forms.

Definition Two one-dimensional primitive forms of the same or different kind are said to be *projective* and their relation is called a *projective correspondence* or *projectivity*, if there exists a one-to-one correspondence between them such that to any harmonic set of four elements in one form, there corresponds a harmonic set of the corresponding four elements in the other.

If A_1, B_1, C_1, ... and A_2, B_2, C_2, ... are corresponding elements of two projective one-dimensional primitive forms $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\mathcal{F}_1$ and $\mathcal{F}_2$, or $A_1B_1C_1$ ... and $A_2B_2C_2$ ... respectively are in a projective relation, or in short, *projective*.

The following theorem derives its name from the fact that it opens the way to many important and characteristic theorems of projective geometry.

Theorem 1.1 (Fundamental Theorem of Projective Geometry) *A projectivity between two one-dimensional primitive forms is uniquely determined by three pairs of corresponding elements.*

The axioms of order and the axiom of continuity **(OC)** are instrumental for the proof of this theorem, as is the concept of harmonic sets (see e.g. Locher [1940] [1957], Coxeter [l960] [1965]).

1.2 One-dimensional collineations and correlations

One-dimensional primitive forms comprise ranges of points, pencils of lines, and pencils of planes. Projective correspondences between two one-dimensional primitive forms can be divided in two classes according to whether they involve two primitive forms of the same or a different kind. In the first case they are called *collineations* and in the second *correlations*.

A projectivity of a one-dimensional primitive form into itself is always a collineation. One then speaks either of two coincident collinear primitive forms, or simply of a collineation. An element that corresponds to itself is called a *self-corresponding*, or *invariant element*.

Theorem 1.2 *A projectivity of a one-dimensional primitive form into itself that has more than two invariant elements is the identity.*

Proof: Take M, N, U as three invariant elements of a given projectivity, say $\boldsymbol{P}$. According to Theorem 1.1, the three pairs of corresponding elements $M = M'$, $N = N'$, $U = U'$ determine a unique projectivity. Since the identity fullfills these conditions, $\boldsymbol{P}$ *is* the identity.

Definition A projectivity of a one-dimensional primitive form into itself is said to be *hyperbolic*, *parabolic* or *elliptic* according to whether the number of proper (real) invariant elements is two, one, or zero, respectively.

It can be said that every projectivity of a one-dimensional primitive form into itself has two invariant elements; in the parabolic case they coincide and in the elliptic case they are *conjugate imaginary*, or *improper*, and can, according to von Staudt [1856], be represented by elliptic involutions (see section 1.4). If invariant elements are mentioned here, this expression means *proper*, i.e. *real*, invariant elements, unless it is stated explicitly that imaginary or improper ones are included.

If A_1, B_1, C_1 are three elements of a one-dimensional primitive Form $\mathcal{F}$ and A_2, B_2, C_2 the corresponding elements of a projectivity of $\mathcal{F}$ into itself, then the sense $A_2B_2C_2$ is either the same as $A_1B_1C_1$ or equal to the sense $A_1C_1B_1$.

Definition A projectivity of a one-dimensional primitive form into itself is called *direct* or *opposite* according to whether it preserves or reverses sense.

Theorem 1.3 *Every opposite projectivity is hyperbolic.*

Proof: See Coxeter [1965], pp. 36–39.

Therefore, every elliptic or parabolic projectivity is direct.

1.3 One-dimensional perspectivities

Definition A projectivity between two one-dimensional primitive forms of a different kind, i.e. a correlation, is called *perspective*, if two elements correspond to each other if and only if they contain each other.

Definition A projectivity between two distinct one-dimensional primitive forms $\mathcal{F}_1$ and $\mathcal{F}_2$ of the same kind, i.e. a collineation, is called *perspective*, if there exists a one-dimensional primitive form $\mathcal{F}$ of different kind (not having its base in common with one of the given forms) that is perspective with both $\mathcal{F}_1$ and $\mathcal{F}_2$.

Theorem 1.4 *A collineation between two distinct one-dimensional primitive forms of the same kind lying in one two-dimensional primitive form is perspective if and only if their common element is self-corresponding.*

Proof: Take M as the self-corresponding element and A, A' and B, B' as two distinct pairs of corresponding elements. The collineation between the two one-dimensional primitive forms is then uniquely determined by the projective relation between MAB and $MA'B'$. In addition, AA' and BB' determine uniquely the base of the primitive form (of a different kind) that is perspective to both of the given collinear primitive forms. – On the other hand, if the collineation between the two primitive forms is a perspectivity, then according to the definition, there exists a one-dimensional primitive form $\mathfrak{F}$ of a different kind that is perspective to both of them. The latter two always have a common element which must therefore be self-corresponding.

Definition If the one-dimensional primitive form that is perspective to both given primitive forms that are related by a perspectivity is a pencil of planes or a range of points, its base line is called the *axis of perspectivity*; if it is a pencil of lines, its center is called the *center of perspectivity*.

However, not every perspective collineation between two one-dimensional primitive forms has a common self-corresponding element. In particular, two skew ranges of points that are perspective to one pencil of planes are perspective to each other and two skew pencils of planes that are perspective to one range of points are also perspective to each other. The same holds true for two pencils of lines in different planes and with distinct centers that are both perspective either to one pencil of planes or range of points.

If A_1, B_1, C_1, ... correspond to A_2, B_2, C_2, ... in a perspectivity between two one-dimensional primitive forms $\mathfrak{F}_1$ and $\mathfrak{F}_2$, then $\mathfrak{F}_1$ and $\mathfrak{F}_2$ or $A_1B_1C_1$ ... and $A_2B_2C_2$ ... repectively are in a perspective relation, or in short, *perspective*.

Another direct consequence of the Fundamental Theorem is the following Theorem 1.5, which is very helpful in constructing corresponding elements of two collinear one-dimensional primitive forms.

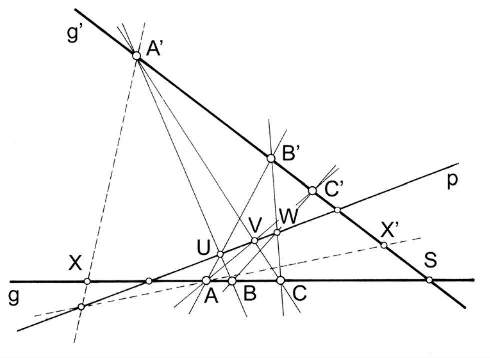

Figure 1.1A

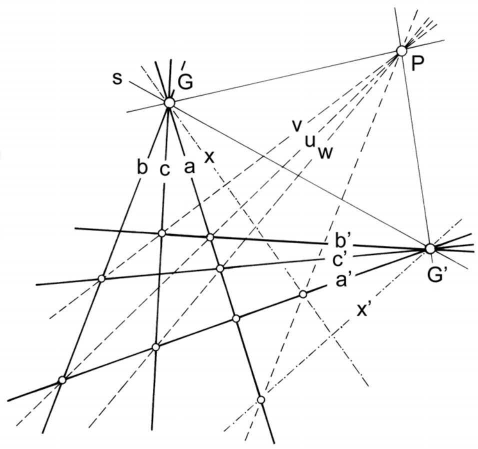

Figure 1.1a

Definition Take A, B, C, ... and A', B', C', ... as corresponding points of two distinct ranges of points g and g' lying in one plane that are related by a collineation. The lines that join a point P of g with any point Q' of g' that does not

Definition Take a, b, c, ... and a', b', c', ... as corresponding lines of two distinct pencils of lines G and G' lying in one plane that are related by a collineation. The intersection point of a line p of G with any line q' of G' that does not cor-

correspond to P is called a *cross-line*; the line that joins the points P', Q that correspond respectively to P, Q' is called the *corresponding cross-line*.

In particular, $AB', A'B$ and $BC', B'C$ are pairs of corresponding cross-lines (Figure 1.1A).

Theorem 1.5A *The intersection points of corresponding cross-lines of two ranges of points in a plane that are related by a collineation all lie on a line.*

Proof: Project the points of g from A' and the points of g' from A. This yields two perspective pencils of lines since AA' is a self-corresponding common line (Theorem 1.4). The axis of perspectivity, p, is then determined by the intersection points U, V of the corresponding cross-lines $A'B, AB'$ and $A'C, AC'$ (Figure 1.1A). – To construct to an arbitrary point X on g its corresponding point X' on g', intersect $A'X$ with p and join this point with A, which finally yields X'. Is $X = pg$, then $X' = gg'$ and if $X = gg'$, then $X' = pg'$. – Hence, in the given collineation, to $gg' = S$ in g corresponds pg' and to S in g' corresponds pg. But this implies that the axis of perspectivity of any two pencils in corresponding points of g, g' (for example in B, B' or C, C') must pass through the same points, namely the points that correspond to $S = gg'$ with respect to g and g'. Hence, the intersection points of all corresponding cross-lines all lie in one line.

respond to p is called a *cross-point*; the point of intersection of the lines p', q that correspond respectively to p, q' is called the *corresponding cross-point*.

In particular, $ab', a'b$ and $bc', b'c$ are pairs of corresponding cross-points (Figure 1.1a).

Theorem 1.5a *The lines joining corresponding cross-points of two pencils of lines in a plane that are related by a collineation all pass through a point.*

Proof: Intersect the lines of G by a' and the lines of G' by a. This yields two perspective ranges of points since aa' is a self-corresponding common point (Theorem 1.4). The center of perspectivity, P, is then determined by the lines u, v joining the corresponding cross-points $a'b, ab'$ and $a'c, ac'$ (Figure 1.1a). – To construct the line x' in G' that corresponds to an arbitrary line x in G, join $a'x$ with P and intersect this line with a, which finally yields x'. Is $x = PG$, then $x' = GG'$ and if $x = GG'$, then $x' = PG'$. – Hence, in the given collineation, to $GG' = s$ in G corresponds PG' and to s in G' corresponds PG. But this implies that the center of perspectivity of any two ranges in corresponding lines of G, G' (for example in b, b' or c, c') must lie on the same lines, namely the lines that correspond to $s = GG'$ with respect to G and G'. Hence, the lines that join corresponding cross-lines all pass through one point.

The corresponding theorems for two pencils of lines or planes in a bundle can easily be derived by projecting the plane figures of Theorem 1.5 from a point outside the plane. For some proofs, the following Theorem 1.6 turns out to be useful.

Theorem 1.6 *Given four arbitrary but distinct elements of a one-dimensional primitive form, there exists a projectivity that interchanges pairs among these four elements.*

Proof: Take for example four points R, S, T, U lying on a line l. Choose a point Z_1 outside l and join Z_1 with R, S and T; select pick a point Z_3 on Z_1S to arrive at the lines $z_1 = Z_3U$ and Z_3R (Figure 1.2). Then one has the following chain of perspectivities: l is perspective to z_1, z_1 is perspective to $z_2 = Z_1T$ and z_2 is perspective to l, with the centers Z_1, $Z_2 = R$ and Z_3 respectively. However, this yields a projectivity between $RSTU$ and $SRUT$ as desired.

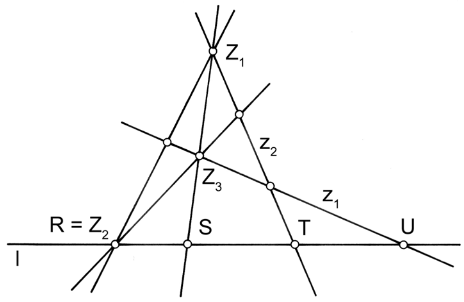

Figure 1.2

1.4 One-dimensional involutions

The concept of involution will be seen to have a fundamental bearing on the subject of non-Euclidean geometry (chapter 3). To begin with, a slightly more general concept is presented, namely the notion of a periodic projectivity.

Definition A projectivity $\boldsymbol{P}$ of a one-dimensional primitive form into itself whose *m-th* power, $m > 1$, is a positive integer, is equal to the identity, $\boldsymbol{P}^m = \boldsymbol{E}$, and is said to be *cyclic* or *periodic*. If $n > 1$ is the smallest number for which this happens, n is called the *period*.

Definition A projectivity of period two is called an *involution*.

In this case $\boldsymbol{P}^2 = \boldsymbol{E}$, and since the inverse $\boldsymbol{P}^{-1}$ exists, the relation $\boldsymbol{P} = \boldsymbol{P}^{-1}$ holds, that is, an involution is equal to its inverse. Henceforth, an involution on a one-dimensional primitive form exchanges its elements in pairs . Such pairs of corresponding elements are said to be *doubly* corresponding. It is remarkable that it is sufficient for a projectivity to be an involution to exchange at least one pair of elements, as the following Theorem 1.7 shows.

Theorem 1.7 *A projectivity of a one-dimensional primitive form into itself that interchanges two elements is an involution.*

Proof: If A, A' is the doubly corresponding pair and X, X' any other corresponding pair, then by Theorem 1.1 the given projectivity is the only one which is determined by the projective relation between $AA'X$ and $A'AX'$. But by Theorem 1.6, there is a projectivity for which $AA'XX'$ and $A'AX'X$ are in a projective relation. Hence, this must be the same as the given projectivity, and X, X' is a doubly corresponding pair. Since X was arbitrary, the given projectivity is an involution.

Theorems 1.1 and 1.7 immediately imply Theorem 1.8.

Theorem 1.8 *An involution is uniquely determined by any two of its doubly corresponding pairs.*

Definition Involutions are called *hyperbolic* or *elliptic* according to whether they have two or no (real or proper) invariant elements.

In particular, there is no parabolic involution:

Theorem 1.9 *If an involution in a one-dimensional primitive form has one invariant element it has another, and any two corresponding elements are harmonic conjugates with respect to the two invariant elements.*

Proof: Take A, A' as any pair in an involution in which M is an invariant element. Then the harmonic conjugate, N, of M with respect to A and A' is also a harmonic conjugate of M with respect to A' and A. But since an involution is a projectivity and hence leaves harmonic sets invariant, N is also an invariant element (different from M, since the points $MANA'$ form a harmonic set, that is, M and N are separated by A and A'). If any other pair X, X' is used instead of A, A', one still obtains the same harmonic conjugate N, since otherwise the involution would have more than two invariant elements.

By Theorem 1.3, every opposite involution is hyperbolic; conversely, it can be shown that every hyperbolic involution is opposite (Coxeter [1965], section 2.7). This yields Theorem 1.10.

Theorem 1.10 *An involution is elliptic or hyperbolic according to whether it is direct or opposite.*

Theorem 1.11 *An involution is elliptic or hyperbolic if and only if two of its corresponding pairs of elements do or do not separate each other.*

Proof: If the involution determined by the pairs A, A' and B, B' is elliptic and thus direct (Theorem 1.10), the sense determined by $AA'B$ is the same than that of $A'AB'$ (but different from the sense $AA'B'$), hence A, A' and B, B' separate each other. Similarly, if the involution is hyperbolic, the sense is opposite (Theorem 1.10) and hence the senses $AA'B$ and $AA'B'$ are equal; consequently A, A' and B, B' do not separate each other. – Conversely, if A, A' and B, B' are separated or not, then from Theorem 1.10 follows that the involution is elliptic or hyperbolic respectively-

Theorem 1.12 *Two involutions in a one-dimensional primitive form, one of which at least is elliptic, always have a common corresponding pair of elements.*

Proof: Assume first that one of the involutions is hyperbolic and the other elliptic. The first is opposite and the latter direct (Theorem 1.10); hence, the composition of these two involutions must be opposite and hence has two invariant elements (Theorem 1.3). Clearly these two invariant elements constitute a common corresponding pair of elements. (For the case in which both involutions are elliptic, see Young [1930], section 17, or Enriques [1915], section 37.)

1.5 Two-dimensional collineations and correlations

The primitive elements of three-dimensional projective space are points, lines and planes. Two primitive elements are called *different*, if they are distinct but of the same kind and *dissimilar* if they are of different kind (whether or not they contain each other is not part of this definition).

The two-dimensional primitive forms are the field of points, the field of lines, the bundle of planes and the bundle of lines (see the introduction). One can classify correspondences between primitive forms that preserve the relation of primitive elements containing each other according to the following definitions.

Definition A one-to-one correspondence between two two-dimensional primitive forms of the same kind is called a *two-dimensional collineation* if to two dissimilar primitive elements a, B containing each other of one primitive form there correspond two dissimilar primitive elements a', B' containing each other of the other primitive form, such that a, a' and B, B' are each pairs of different primitive elements.

Hence, in a collineation, ranges of points or pencils of planes correspond to ranges of points or pencils of planes, respectively, and pencils of lines correspond to pencils of lines such that the relation of primitive elements containing each other is preserved.

In particular, if one adds in a similar fashion the collineation between a field and a bundle, which is not covered by the definition above, the following cases of *two-dimensional collineations* exist.

(i)	Field-Field:	Points correspond to points and lines to lines.
(ii)	Field-Bundle:	Points of the field correspond to lines of the bundle and lines of the field correspond to planes of the bundle.
(iii)	Bundle-Bundle:	Lines correspond to lines and planes to planes.

Definition A one-to-one correspondence between two two-dimensional primitive forms of the same kind is called a *two-dimensional correlation* if to two dissimilar primitive elements a, B containing each other of one primitive form there correspond two dissimilar primitive elements A', b' containing each other of the other primitive form, such that a, A' and B, b' are each pairs of dissimilar primitive elements.

Hence, in a correlation, ranges of points or pencils of planes correspond to pencils of planes or ranges of points respectively, and pencils of lines correspond to pencils of lines such that the relation of primitive elements containing each other is preserved.

In analogy to the three cases of collineations, one has, in particular, the following three cases of *two-dimensional correlations*. Note again that the correlation between a field and a bundle is not covered by the definition above, but fits naturally into this context. (As an exercise, the reader may modify the above definitions such that they encompass the collineations and correlations between field and bundles.)

(i)	Field-Field:	Points correspond to lines and lines to points.
(ii)	Field-Bundle:	Points of the field correspond to planes of the bundle and lines of the field correspond to lines of the bundle.
(iii)	Bundle-Bundle:	Lines correspond to planes and planes to lines.

One can still handle collineations or correlations between two bundles or two fields as correspondences between different bundles or fields when the latter coincide. However, it is sometimes useful to refer to them as collineations or correlations of bundles or fields into themselves, in other words, as *autocollineations* or *autocorrelations* of bundles or fields (see section 1.6).

The extension of the concept of projectivity from the one-dimensional case to higher-dimensional ones goes as follows.

Definition A one-to-one correspondence between two two-dimensional primitive forms is called a *two-dimensional projectivity*, if it preserves the relation of primitive elements containing each other, or, equivalently, if one-dimensional primitive forms correspond to one-dimensional primitive forms such that they are projective.

The equivalence stated in this definition is due to the fact that harmonic sets of four elements are defined in terms of quadrangles and quadrilaterals (and their counterparts in bundles). Since collineations and correlations preserve the relation of primitive elements containing each other and encompass all possible cases of such correspondences, one concludes that they constitute exactly the set of all possible projectivities. Projectivities between two-dimensional primitive forms are called *two-dimensional projectivities*.

Definition Three primitive elements of the same kind are called *linearly independent* if they do not belong to one one-dimensional primitive form. Four primitive elements of the same kind belonging to a two-dimensional primitive form are called *linearly independent* if no three of them lie in one one-dimensional primitive form.

With the foregoing material, the *Fundamental Theorem of Two-dimensional Projective Geometry* can now be stated as follows (for a proof, see Coxeter [1965], pp. 49–52, or Locher [1940], pp. 195–197).

Theorem 1.13 *A two-dimensional projectivity exists and is uniquely determined if the elements of a set of four linearly independent primitive elements in one two-dimensional primitive form correspond uniquely to the elements of a set of four linearly independent primitive elements in the other two-dimensional primitive form.*

For example, a collineation or correlation between two fields is uniquely determined by two corresponding quadrangles, or a quadrangle corresponding to a quadrilateral respectively.

Theorem 1.14 *A two-dimensional projectivity exists and is uniquely determined if two pairs of different one-dimensional primitive forms each lying in a two-dimensional primitive form are projective such that the common element in one pair corresponds to the common element in the other.*

For example, two different pencils of lines, say, within a plane α which are projective to two different ranges of points in a plane β, uniquely determine a correlation between these two fields α and β, if the common line of the two pencils of lines corresponds to the common point of the two ranges of points. In fact, Theorem 1.14 is equivalent to Theorem 1.13. This is easy to see and will be left as an exercise to the reader.

1.6 Projectivities of fields or bundles into themselves: Two-dimensional autocollineations

Projectivities between a field and a bundle give rise to projectivities of the field or bundle into themselves if one allows the field and the bundle to intersect or project each other respectively. In addition, two projective fields or bundles may coincide, thus producing a projectivity of these forms into themselves. According as these projectivities are collineations or correlations, they are called *two-dimensional autocollineations* or *two-dimensional autocorrelations* respectively. Autocorrelations will be treated in section 5.2.

The following Theorem 1.15A is equivalent to the uniqueness property stated in the Fundamental Theorem of Two-dimensional Projective Geometry, Theorem 1.13.

Theorem 1.15A *If a collineation of a field into itself leaves four linearly independent points or lines invariant, the collineation is the identity.*

Theorem 1.15a *If a collineation of a bundle into itself leaves four linearly independent planes or lines invariant, the collineation is the identity.*

Proof: Consider four linearly independent elements in a field (or in a bundle); according to Theorem 1.13, there is a unique projectivity $\boldsymbol{P}$ which transforms each of these elements into itself. But the identity has the same property, hence $\boldsymbol{P}$ *is* the identity.

In order to derive the uniqueness property stated in Theorem 1.13 from Theorem 1.15, consider two corresponding sets, Q_1 and Q_2, of four linearly independent elements in a plane (or in a bundle). Then, according to the existence property stated in Theorem 1.13, there are two projectivities $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ such that $\boldsymbol{P}_1(Q_1) = Q_2$ and

$\boldsymbol{P}_2^{-1}(Q_2) = Q_1$. Since the projectivity $\boldsymbol{P}_1\boldsymbol{P}_2^{-1}(Q_2) = Q_2$, $\boldsymbol{P}_1\boldsymbol{P}_2^{-1}$ leaves the set Q_2 invariant, it must be the identity according to Theorem 1.15, or equivalently, one must have $\boldsymbol{P}_1 \equiv \boldsymbol{P}_2$.

One concludes from Theorem 1.15 that a collineation of a field or a bundle into itself that is not the identity has at most three linearly independent invariant elements. But there is also a lower bound for the set of invariant elements.

Theorem 1.16A *Every autocollineation of a field [bundle] has at least one proper invariant point [line].*

Theorem 1.16a *Every autocollineation of a bundle [field] has at least one proper invariant plane [line].*

Proof (left side): Coxeter [1965], p. 62; Enriques [1915], p. 265f.

The following theorems give some more information as to how the invariant elements are arranged; they will help to find all possible classes of two-dimensional autocollineations.

Theorem 1.17A *If an autocollineation of a field has an invariant point, then it has an invariant line, and conversely.*

Theorem 1.17a *If an autocollineation of a bundle has an invariant plane, then it has an invariant line, and conversely.*

Proof: This is an immediate consequence of the principle of duality (for a direct proof, see Enriques [1915], p. 149f., or Reye [1907], p. 70).

Theorem 1.18A *If a non-identical autocollineation of a field has the points of a range as invariant elements, then it has also the lines of a pencil as invariant elements, and conversely.*

Theorem 1.18a *If a non-identical autocollineation of a bundle has the planes of a pencil as invariant elements, then it has also the lines of a pencil as invariant elements, and conversely.*

Proof (left side): Consider a non-identical autocollineation of a field. Let z be the base-line of a range of invariant points. A pair of corresponding lines l and l' will meet in a self-corresponding point $P = P'$ (Figure 1.3), hence l and l' are perspective (Theorem 1.4) and lines connecting corresponding points of l and l', e.g. A, A' and B, B', will pass through some point Z that may lie on z. All lines through Z are invariant, since they contain an invariant point of z and a pair of corresponding points lying on l and l'. –

Conversely, the existence of a pencil of invariant lines implies the existence of two perspective pencils of lines (the centers of which lie on an invariant line) and hence the existence of an invariant range of points.

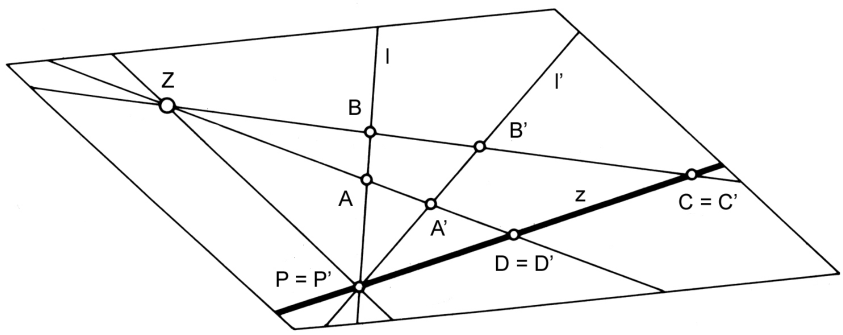

Figure 1.3

1.7 Classification of two-dimensional autocollineations

In the following, a complete *classification of two-dimensional autocollineations* is given. Only the planar field will be discussed, since the autocollineations of a bundle can be derived from them by projecting the field from a point outside. To begin with, assume that the collineation is not the identity and that there exists an invariant point E_1 (Theorem 1.16) and an invariant line e_1 (Theorem 1.17) which does not pass through E_1. In this case, the projectivity induced on e_1 is either hyperbolic, elliptic, parabolic, or the identity, as one knows from one-dimensional projective geometry (section 1.2); hence one has the following classes **(Ia)**, **(Ib)**, **(II)**, **(III)** (Figure 1.4).

Class* (Ia): *General collineation

Invariant points:
E_1, E_2, E_3
Invariant lines:
e_1, e_2, e_3

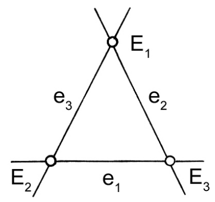

Figure 1.4 (Ia)

***Class* (Ib)**

Invariant point:
E_1
Invariant line:
e_1

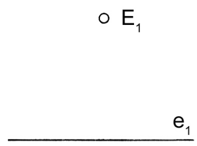

Figure 1.4 (Ib)

***Class* (II)**

Invariant points:
E_1, E_2
Invariant lines:
e_1, e_3

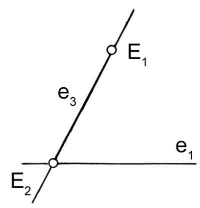

Figure 1.4 (II)

Class* (III): *Homology

Invariant points:
range of points e_1
Invariant lines:
pencil of lines E_1

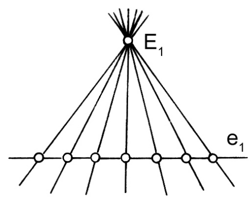

Figure 1.4 (III)

If all invariant points are contained in all invariant lines, then there is either more than one invariant line, or exactly one. In the first case, there exist at least two invariant lines and Theorem 1.17 implies that there are also at least two non-coincident invariant points; but the only invariant point which is contained in both invariant lines is their intersection point, hence this case is impossible. In the second case, the collineation on the invariant line can only be parabolic or the identity, since two non-coincident invariant points are impossible (they would imply more than one invariant line, in contradiction to the assumption). Together with Theorem 1.18, this implies the following additional classes **(IV)**, **(V)** (Figure 1.4).

***Class* (IV)**
Invariant point:
E_2
Invariant line:
e_1

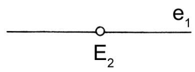

Figure 1.4 (IV)

Class* (V): *Elation
Invariant points:
range of points e_1
Invariant lines:
pencil of lines E_2

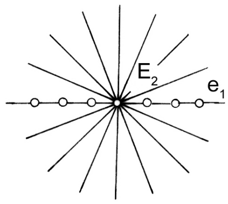

Figure 1.4 (V)

In addition, there is the identity:

Class* (VI): *Identity
Invariant points:
field of points
Invariant lines:
field of lines

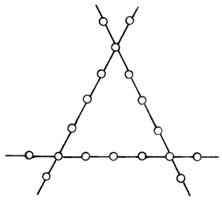

Figure 1.4 (VI)

Hence one concludes that there are seven classes of two-dimensional autocollineations. The proofs of existence for the classes **(Ib)**, **(II)**, **(IV)** will be left to the reader. The classes **(III)** and **(V)** will be discussed without details in section 1.8.

1.8 Two-dimensional perspective collineations

Definition A collineation between a field and a bundle is called *perspective* if all pairs of corresponding elements contain each other.

According to Theorem 1.13, this is the case if it is true for at least four linearly independent elements. More interesting cases arise when collineations between two fields or two bundles are considered.

Definition A collineation between two fields or two bundles is called *perspective*, or a *perspectivity*, if there is at least one one-dimensional primitive form the elements of which are self-corresponding, that is, invariant.

Theorem 1.19A *If a collineation between two fields (that may coincide) has the points of a range as invariant elements, then it is perspective and the lines and planes of all joins of pairs of corresponding points and lines respectively pass through one point.*

Theorem 1.19a *If a collineation between two bundles (that may coincide) has the planes of a pencil as invariant elements, then it is perspective and the lines and points of intersection of all pairs of corresponding planes and lines respectively lie in one plane.*

The proof of Theorem 1.19A for a perspective collineation between the fields ε and ε' will be left as an exercise (see Figure 1.5).

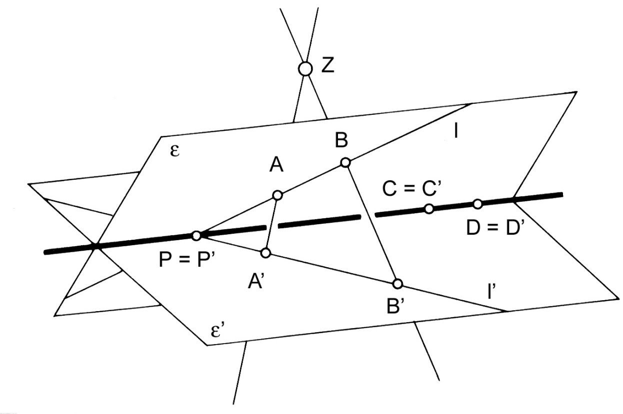

Figure 1.5

With respect to a perspective collineation between two planes, the unique invariant point Z is called the *center of perspectivity* and the unique invariant range z the *axis of perspectivity*. In the case of a perspective collineation between two bundles, there is a unique *axis* and a unique *axial plane*. Perspective autocollineations are classified into *elations* or *homologies* according to whether the center respectively does or does not lie on the axis or axial plane.

The easiest way to determine a perspectivity is expressed in the following Theorem 1.20, the proof of which is also left to the reader.

Theorem 1.20A *A perspective collineation between two plane fields is uniquely determined by its center, its axis and a pair of corresponding elements.*

Theorem 1.20a *A perspective collineation between two bundles is uniquely determined by its axial plane, its axis and a pair of corresponding elements.*

1.9 Two-dimensional involutions

Definition *Involutions* are projectivities of fields or bundles into themselves with period two, in other words, projectivities which are equal to their own inverse.

In this case the elements of the plane or the bundle respectively are ordered in *doubly corresponding* or *involutory* pairs.

Definition A projectivity between a bundle and a field is called *involutory*, if the section of the bundle with the field or the projection of the field from the bundle generates an involution in the field or in the bundle respectively.

1.9.1 *Two-dimensional harmonic reflections*

A simple example of an involutory collineation is the involutory perspectivity. Let a perspectivity in a plane be given by a center Z, an axis z, and a pair of corresponding points Q, Q' such that $ZQPQ'$ forms a harmonic set, where $P = P'$ is the point of intersection of QQ' with z (Figure 1.6). Since harmonic sets correspond to harmonic sets, one has a projective relation between $ZQPQ'$ and $ZQ'PQ$, and hence the perspectivity is involutory. It is called a *harmonic homology* or a *harmonic reflection*. As it turns out, this is the only two-dimensional involutory collineation.

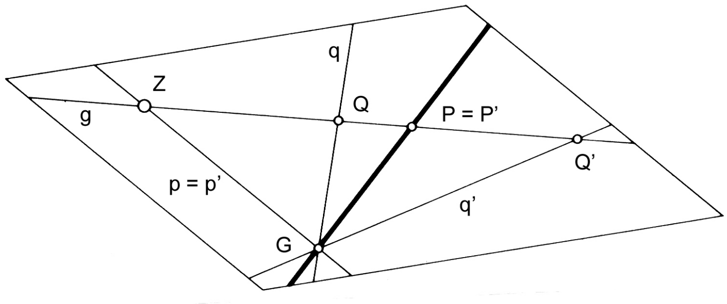

Figure 1.6

Theorem 1.21 *Every two-dimensional involutory collineation is a harmonic homology.*
Proof: Coxeter [1955], p.64; Locher [1940], pp. 205f.

1.9.2 *Two-dimensional polarities*

Definition Involutory correlations of fields or bundles into themselves are called *polarities*. Corresponding points and lines, or lines and planes respectively, are called *poles* and *polars*.

For more details on two-dimensional polarities, see section 3.3.

1.10 Notes and references

For more information on one- and two-dimensional projective geometry, see Coxeter [1955], Meserve [1955], Cremona [1885], Locher [1940][1957], Reye [1909], Veblen/Young [1910], Young [1930], Ostheimer/Ziegler [1996], Bernhard [1984], Edwards [1985].

The synthetic theory of imaginary or complex elements starts with von Staudt [1856][1857][1860]. For discussions of this theory, see Coolidge [1924], Juel [1934], Locher [1940][1970], Reye [1909].

The classification of two-dimensional autocollineations can also be be found in Baldus [1928], Newson [1897] [1899]. For an algebraic treatment on the basis of path curves, see Boer [2004]. A history of elementary projective geometry can be found in Kötter [1901].

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