Selected Topics in Three-Dimensional Synthetic Projective Geometry

Introduction, References, and Index

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Preface

The following chapters are a contribution to visual geometry, mainly three-dimensional projective geometry and line geometry. The development of flexible geometric thinking seems to be somewhat neglected nowadays in the training of mathematicians and scientists in general. But the need for engineers and scientists who feel comfortable with the subtleties of spatial geometry is apparent, even in spite of the now available computer-aided methods of graphic design. And there may be no better way to get deeply involved in spatial geometry than in exercising the skills of visual representation by the means of synthetic geometry. Thus I have adopted the *synthetic method,* which forms a characteristic feature of this presentation.

Some experts identify the synthetic method with the axiomatic method, but I would rather emphasize that the synthetic method deals directly with the properties of the geometric objects themselves without representing them in terms of other objects, for example, in algebraic equations. Of course, one often arrives much faster at geometric results or propositions by applying some kind of algebraic symbolism, but, during the process, the geometric object in question can disappear very easily from our imagination. – This approach might be unfamiliar to many readers, but I know of nothing else that trains the geometric intuition more effectively than synthetic geometry.

Another characteristic feature of this presentation is the approach to Euclidean geometry, which is not conceived as a primitive space form but is consistently treated as a certain kind of limiting case of general projective metric geometry. Not only can this be of practical importance, but it will particularly refine and clarify the logical skills and the basic concepts of geometry. In addition, it helps one keep an open mind with respect to new ideas and techniques which might be of special significance in the future.

The reader is expected to be familiar at least with the fundamentals of plane projective geometry. Since there are many good textbooks on this subject in the English and German literature (see the references), it is not difficult to fill in the necessary details and supplementary information.

I discuss many topics that have never been treated comprehensively elsewhere in the English literature. In particular, this includes the synthetic treatment of quadratic surfaces, the complete classification and subsequent discussion of collineations in three-dimensional projective space, as well as the discussion of congruences and their relation to cubic space curves.

The theory of complex elements, or imaginary elements as they are called in this text, is not treated in any systematic way. There are occasional remarks for the informed reader but no comprehensive account of the theory of imaginary points, lines or planes, or of imaginary quadratic surfaces. This would call for a special treatise which goes well beyond the level and scope I intended for this text.

There are no formal exercises but the reader is encouraged again and again to carry out details in proofs, think about special cases of general theorems, dualize theorems as well as proofs, and look for affine and Euclidean (or even polar-Euclidean) specializations. This provides enough material to think about and to get the taste and practice of doing synthetic projective geometry. In addition, the reader would do well to make his or her own drawings as much as and as often it seems appropriate.

This presentation of three-dimensional projective geometry starts with a short introduction to fundamental concepts and methods of projective geometry. Chapter 1 provides an account of two-dimensional projective geometry which is supposed to serve as an introductory reminder rather than a comprehensive overview. Many theorems are mentioned without proofs and some important topics are not discussed.

Chapter 2 contains a discussion of three-dimensional projectivities including their complete classification. Chapters 3 and 4 cover the theory of curves and cones as well as quadratic surfaces and polarities. In chapter 5, the theory of involutory collineations and correlations is worked out in detail; these involutions give rise to some important families of lines.

Chapter 6 covers the foundations of projective metric geometry which are used throughout the text to specialize general projective properties to metric ones, in particular, to Euclidean properties. This chapter contains only the material needed for this specialization; there is, of course, much more to be said about non-Euclidean geometry.

Some fundamental notions of line geometry are introduced in chapter 7. Chapters 8 and 9 give a rather complete treatment of the most important linear families of lines, namely linear complexes and linear congruences. Chapters 10, 11 and 12 treat families of lines generated by collineations between primitive forms. This includes as special cases linear congruences, but goes beyond them to congruences of higher order and class which are connected with cubic space curves. Chapter 12 culminates in the proof of the existence of at least one proper (real) invariant line for any collineation in three-dimensional projective space. Chapter 13 covers non-Euclidean, affine and Euclidean properties of linear families of lines.

Chapters 14, 15 and 16 go deep into the geometry of linear complexes and their linear systems of lines within three-dimensional projective space, culminating in chapter 17 with the introduction of the five-dimensional linear manifold of linear complexes.

This text, published here as series of papers, can serve at least two purposes: to refresh the memory of someone familiar with projective geometry concerning the beautiful world of three-dimensional projective geometry and enhance his or her imagination, and to get oneself involved again in thinking it all through once more. On the other hand, someone not familiar with this kind of geometry is provided with guidance right into the heart of the matter, or rather, the heart of the «projective spirit». From there one can start to delve into more detailed or specialized literature (see the notes and reference section at the end of each chapter) and/or consider some applied work, for example, by George Adams [1977][1996], Peter Gschwind [1989][1991][2000][2004], Lawrence Edwards [1986] [1993], Nick Thomas [1999].

History of the text and acknowledgements

Originally, this text was intended as a translation of my *Synthetische Liniengeometrie* [1981]. In working through my book again, a rough translation made by Amos Franceschelli (Spring Valley, New York) was very useful, but I soon realized that I had to rewrite most parts of it. Finally, a new text emerged. It was Joseph Duffy, Director of the «Center for Intelligent Machines and Robotics» at the University of Florida in Gainesville, who encouraged me to undertake the laborious process of rewriting my old book and provided me with the financial resources to complete the major parts of it while working in his Center during the academic year 1985/86. The manuscript was nearly completed during a

research sojourn in the Department of Physics at Arizona State University in Tempe 1986/87. A final version was produced in my first year 1987/88 at the Mathematical-Astronomical Section at the Goetheanum in Dornach, Switzerland, but has never been published.

Many thanks go to my students and colleagues at the University of Florida, as well as to Peter Gschwind and Arnold Bernhard, who provided a warm and resourceful audience for expounding my thoughts. Last but not least, this text owes much to the spirit and work of Theodor Reye (1839–1919), George Adams (1894–1963), and Louis Locher (1906–1962), none of whom did I ever meet in person, but who nevertheless deeply influenced my own thinking through their work (for biographical and bibliographical information on Adams and Locher, see Ziegler [2001]).

Peter Gschwind, the new editor of the «Mathematisch-Physikalische Korrespondenz» since 1999, provided me with the opportunity to make this manuscript available in print as a series of papers. In reworking my text thoroughly beginning in 2003 up to 2005, I rewrote and rearranged some parts to make it intuitively more accessible and added some references to the recent literature.

Considering the vast amount of work that has been done already in the late 19th and early 20th century in the field of three-dimensional projective geometry, there is essentially nothing new in this text. However, some topics have never been treated thoroughly and systematically within the mathematical literature inspired by Rudolf Steiner′s spiritual science or anthroposophy, particularly not in the English language (with one partial exception: Edwards [1985]). These topics include the complete classification of projectivities in three-dimensional projective space (chapter 2), the projective theory of quadratic surfaces in three-dimensional projective space (chapter 4), and, in particular, non-linear families of lines generated by collineations between bundles and fields (chapter 10), including twisted cubics and cubic developables (chapter 11), and collineations in three-dimensional projective space that generate tetrahedral quadratic complexes (chapter 12).

Many thanks are also due to Alec Schaerer and Marius Oosterveld for producing a rough typescript from the original manuscript that was partly written by typewriter and corrected by hand. Christoph Jäggy helped me with scanning the figures, some of which were taken from my «Liniengeometrie», but many were drawn by hand especially for this new text. The English proofreading has been done by John O'Brien, which is very much appreciated. In addition, Lou de Boer was kind enough to check the text mathematically; he contributed valuable suggestions for improving the text which resulted in some major revisions. However, all remaining errors and vague formulations are in the sole responsibility of the author.

Preview of following chapters

1. Projectivities between primitive forms of one and two dimensions
2. Projectivities in three-dimensional space
3. Introduction to curves and surfaces in three-dimensional projective space
4. Surfaces of the second degree in three-dimensional projective space
5. Involutory collineations and polarities in three-dimensional projective space
3. Foundations of three-dimensional Euclidean and non-Euclidean geometry
7. Fundamental notions of line geometry in three-dimensional projective space
8. Linear complexes of lines in three-dimensional projective space
9. Linear congruences of lines in three-dimensional projective space
10. Families of lines in three-dimensional projective space generated by collineations between bundles and fields

11. Twisted cubics and cubic developables in three-dimensional projective space
12. Collineations in three-dimensional projective space: tetrahedral quadratic complexes
13. Non-Euclidean, affine and Euclidean properties of linear families of lines in three-dimensional projective space
14. Reciprocal linear complexes in three-dimensional projective space
15. Linear systems of linear complexes in three-dimensional projective space
16. Three-systems of linear complexes and their induced polarities in three-dimensional projective space
17. The five-dimensional linear manifold of linear complexes

Preliminaries

1. *Axioms and primitive elements*

In three-dimensional synthetic projective geometry the primitive undefined elements are «points», «lines» and «planes» which are related by the axioms in terms of containing each other as well as according to some notions of «order» and «continuity». «Containing each other», particularly «passing through» and «lying in» as well as «order» and «continuity» will not be defined explicitly here, since they are determined implicitly by the axioms. These concepts are thought to have intuitive significance and thus provide justification for the endeavors in which I am about to embark. In other words, I think that it is important that there is some interpretation in which the axioms are seen to be true – otherwise one would be playing a meaningless game.

Following Locher [1940], the axioms for three-dimensional projective geometry can be formulated in a form that clearly reflects their intuitive origins but leave nothing to be desired as to their logical strictness and completeness (see Appendix 2, 3 and 4). However, I am not concerned about their independence since this does not add anything to their understanding but merely complicates the path of reasoning. For strict axiomatic treatments of projective geometry that deal with such problems as consistency, completeness, categoricalness and independence I must refer to the literature, for example Veblen/Young [1910] or Forder [1927]. An excellent axiomatic treatment of two-dimensional projective geometry can be found in Coxeter [1955], which provides a valuable supplement to the approach presented in this series of papers.

There is no point in repeating elementary details here, since our main goal is not so much the treatment of the axiomatic foundations of projective geometry as the exploration of some of the implications of these axioms in three-dimensional projective space. In other words, I must assume the reader to be familiar with elementary projective geometry and thus willing to follow me without hesitation through a quick review of one-dimensional and two-dimensional projective geometry in order to delve more completely into the figures and transformations of three-dimensional projective space.

The *primitive forms* of three-dimensional projective geometry encompass the range of points, the pencil of lines, the pencil of planes, the field of points, the field of lines, the bundle of planes, the bundle of lines and the pencil of lines (Appendix 1). They represent all cases in which each of the *primitive elements,* point, line and plane, are generated by a totality of infinitely many primitive elements of different kinds. The underlying primitive elements of such a primitive form are called its *base* or *bearer*, or, in the case of bundles and pencils of lines, its *center*. For example, the base or bearer of a range of points is the line which contains all the points of the range, etc. With one exception, the pencil of lines, all primitive forms have exactly either one base or one center.

Primitive forms are classified according to their *dimension* (see Appendix 1): the primitive forms of the first dimension are: range of points, pencil of lines and pencil of planes; the primitive elements of the second dimension are: field of points, field of lines, bundle of planes and bundle of lines.

Projection and *intersection* have the usual meaning. For example, a point outside a range of points projects this range in a pencil of lines; a pencil of planes intersects a line in a range of points; a bundle of lines intersects a plane in a field of points, etc.

2. *The principle of duality in three-dimensional projective geometry*

An inspection of the axioms of three-dimensional projective geometry as formulated in Locher [1940] reveals a perfect symmetry with respect to points and planes (see Appendix 2, 3 and 4). In particular, to every statement about points, lines and planes and their relations corresponds a statement about planes, lines and points, respectively. If one employs only the expression «incident» or «in common» to describe the various relationships between primitive elements, one could directly transform a given statement into another valid statement by exchanging the words «plane» and «point». But for the sake of concrete expressions I use the terms «pass through», «lies in», etc. which have to be exchanged accordingly. Statements that are related in such a way that they can be transformed into each other by the procedures described above are called *dual* (with respect to three-dimensional projective geometry).

Given the self-dual character of the set of axioms, the *principle of duality* asserts that to every theorem derived from these axioms there corresponds a dual theorem that can be proved by dualizing the original proof, that is, rewording it in accordance with the exchange of «point» and «plane». Strictly speaking, the principle of duality is not a theorem of the formal theory of projective geometry, if by «formal theory» one means only the statements that can be derived from the axioms. The principle of duality does not involve only points, lines, planes and relations, but, in fact, it is a theorem about theorems, provability etc., in other words, a metatheorem. This was, for example, pointed out by Lakatos [1978] in the context of demonstrating the fruitfulness of informal arguments for enriching a given formal theory.

The principle of duality in three-dimensional projective geometry implies similar principles for two-dimensional projective geometry in a plane or in a point. The resulting relationships give rise to a host of beautifully interconnected theorems and figures (for some details, see Locher [1957], p. 25).

Typographically, dual theorems are presented in double columns. In order to save space and avoid repetition, however, dual statements are not made explicitly in full at some places, but only indicated by some key expressions in square brackets which are to replace the immediately preceding ones.

The true simplicity of the principle of duality misleads many geometers to concentrate on only one aspect of projective geometry, namely the statements in which points play a dominant role. This bias is a consequence of our initial training in Euclidean geometry. As a consequence, the expanding of the dual theory was badly neglected with the excuse that it involves only a mechanical translation process. This is absolutely correct from a *formal* point of view, but does not give credit to the full experience and appreciation of the content of projective geometry. Anyone who tries to grasp intuitively the dual of a given statement concerning point configurations experiences serious difficulties in elevating his imaginative mind beyond formal insight into the bare logical structure. After some training, however, the effort is well rewarded by the ethereal beauty which reveals itself in the realm of the «geometry of planes» in contrast to the more worldly beauty of the «geometry of points» to which one is accustomed since elementary and high school days.

It is for this reason that I have not hesitated to include in most (though not all) places the dual statements as well. I want the reader to have constantly in mind the inherent symmetry of projective geometry based on the principle of duality. And I wish to emphasize that this symmetry is not merely a trivial feature of the formal structure but something that is worth studying in itself, thus leading possibly to new forms of experience.

3. *Order and continuity*

The axioms of order and continuity (see Appendix 4) specify the arrangement of the infinitely many elements of a one-dimensional primitive form. There are many different approaches for the characterization of order and continuity, depending on what concepts are taken as primitive, or undefined. Here the system of axioms developed by Locher [1952] is adopted, since it is very intuitive and particularly well suited to the foundations of synthetic geometry and synthetic kinematics. The primitive concepts are the *move* of an element within a one-dimensional primitive form and the *order* of a finite number of elements, that is, their numbering.

The axioms of order and the axiom of continuity **(OC)** are summarized below in Appendix 4. Based on Axiom (i), *sense* is defined as follows.

Definition If *A, B, C* are three elements of a one-dimensional primitive form and if a move starting at *A* passes the elements *B* and *C* in the order *BC,* then the move has the *sense ABC*.

Axiom (ii) tells that for any finite set of elements there is a unique order or numbering; it is called the *natural order* of this set with respect to the starting element *A* and the sense *ABC*.

Definition If *ABXC* is in natural order with respect to the sense *ABC*, then *X* is said to be *between B* and *C* with respect to *A*, or, equivalently, *B* and *C* are said to be *separated* by *A* and *X*.

Definition and Theorem 1 *Two elements A, B of a one-dimensional primitive form separate the set of all elements of this form into exactly two* segments*; each element X in one segment is separated from each element in the other segment with respect to A, B.*

Proof: Let *C* be an element of the given primitive form which does not coincide with *A* or *B*, and let any element move through this form starting from *A* with the sense *ABC*. All elements, except *A*, that are passed before one arrives at *B* form the one segment; all remaining elements, except *B*, the other segment. If *X* is an element of the first and *Y* an element of the second segment, then *AXBY* represents a natural order, hence *A, B* and *X, Y* are separated.

The two segments of a one-dimensional primitive form determined by two of its elements *A* and *B* are denoted by [*AB*] and]*AB*[.

Definition Consider a segment of a one-dimensional primitive form having a sense that contains an infinite sequence of elements P_1, P_2, P_3, P_4, ... such that every P_i lies before P_{i+1}. Such a sequence is called *monotone*. A *limit element L* of such a sequence is an *element L* such that between P_k and *L* with respect to P_{k-1} lie infinite many elements for every $k > 1$.

Theorem 2 *Every monotone sequence of elements within a segment of a one-dimensional primitive form has exactly one limit element.*

Proof: Take P_1, P_2, P_3, ... as a monotone sequence of elements within a segment [*AB*] such that $AP_1P_2 \ldots B$ is in natural order. This sequence implies a separation of the elements of the primitive form in two sets $\mathcal{X}$, $\mathcal{Y}$ such that every element of [*AB*] that lies before any element of the sequence constitute the set $\mathcal{X}$ and the remaining elements of the primitive form, except *A,* constitute the set $\mathcal{Y}$. Then Axiom (vi) implies the existence and uniqueness of an element *L* within $\mathcal{X}$ or $\mathcal{Y}$ such that *AXLY* is in natural order for every $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Since]*AB*[does only contain elements of $\mathcal{Y}$, *L* is either identical with *B* or lies within [*AB*]. In either case, since all P_i's lie in $\mathcal{X}$, *L* must be a limit element.

Appendix

1. ***Primitive elements and primitive forms of three-dimensional projective geometry***

The expression «infinitely many» in the formulation of the axioms below needs further specification. Fields, for example, contain infinitely many points and lines; but they also contain infinitely many ranges of points and pencils of lines. Obviously, «infinite» is used here with two slightly different meanings. To clarify this, by definition, the primitive forms are classified according to their *dimension*. For this purpose, the symbol «∞^n», n = 0, 1, 2, 3, …, is introduced to represent the «degree of freedom» of the number of primitive elements contained in them. For example, fields of lines are primitive forms of two dimensions and contain ∞^2 lines. In section 7.1 the subject of primitive forms is discussed from a different perspective and studied in more detail: how they are interrelated with each other, as well as how they give rise to higher-dimensional primitive forms.

Definition *Primitive elements and primitive forms of three-dimensional projective geometry*

	Dimension	*«Number» of primitive elements*
Primitive elements		
point	0	$\infty^0 = 1$
line	0	$\infty^0 = 1$
plane	0	$\infty^0 = 1$
Primitive forms		
range of points	1	∞^1
pencil of lines	1	∞^1
pencil of planes	1	∞^1
field of points	2	∞^2
field of lines	2	∞^2
bundle of planes	2	∞^2
bundle of lines	2	∞^2
space of points	3	∞^3
space of planes	3	∞^3
space of lines	4	∞^4

2. ***Axioms for primitive elements containing each other*** **(I)** ***for three-dimensional projective geometry***

In a line lie infinitely many points: the line appears as *range of points*.

In a plane lie infinitely many points: the plane appears as *field of lines*.

In a plane lie infinitely many lines: the plane appears as *field of lines*.

Through a point lying in a plane pass infinitely many lines which lie in the plane, forming a *pencil of lines*.

If a line lies in a plane, then every point of the line (as a range of points) lies in this plane.

Through a line pass infinitely many planes: the line appears as *pencil of planes*.

Through a point pass infinitely many planes: the point appears as *bundle of planes*.

Through a point pass infinitely many lines: the point appears as *bundle of lines*.

In a plane passing through a point lie infinitely many lines which pass through the point, forming a *pencil of lines*.

If a line passes through a point, every plane of the line (as a pencil of planes) passes through the point.

If a point lies in a plane, not every line of the point (as a bundle of lines) lies in the plane.

If a plane passes through a point, not every line in the plane (as a field of lines) passes through the point.

If a point lies in a line, not every plane of the point (as a bundle of planes) passes through the line.

If a plane passes through a line, not every point of the plane (as a field of points) lies on the line.

3. ***Axioms of connection and axioms of intersection* (CI)**
for three-dimensional projective geometry

Two distinct points have exactly one line in common which passes through both points: the *join* of the two points.

Two distinct planes have exactly one line in common which lies in both planes: the *intersection line* of the two planes.

A point and a line not passing through it have exactly one plane in common which passes through the point and the plane: the *connecting plane* of the point and the line.

A plane and a line not lying in it have exactly one point in common which lies in the plane and in the line: the *intersection point* of the plane and the line.

Two distinct lines have either exactly one *intersection point* (which lies in both) and exactly one *connecting plane* (which passes through both) or neither an intersection point nor a connecting line. In the last case they are called *skew*.

4. ***Axioms of order and axiom of continuity* (OC)**
for three-dimensional projective geometry

(i) For every move of an element within a one-dimensional primitive form starting with an element A there exists an opposite move starting with A with the following properties: If the first move passes two distinct elements B, C in the order BC, then the opposite move passes them in the order CB.

(ii) If the starting element A and the sense ABC of a move are given, then the order of any finite number of elements which are passed by this move is uniquely determined.

(iii) If A, B, C are three elements of a one-dimensional primitive form, then there are infinitely many elements X that are between B and C with respect to A.

(iv) If $AA_1A_2 \ldots A_kBB_1B_2 \ldots B_lCC_1C_2 \ldots C_m$ is a set of elements in natural order with starting element A and sense ABC, then $BB_1B_2 \ldots B_lCC_1C_2 \ldots C_mAA_1A_2 \ldots A_k$ is the natural order of these elements with respect to the starting element B and sense BCA.

(v) The natural order of any set of elements of a one-dimensional primitive form is invariant under projection and intersection.

(vi) *Axiom of continuity*: Let A be an arbitrary element of a one-dimensional primitive form. Let the totality of the remaining elements of the primitive form be separated into two sets $\mathcal{X}$ and $\mathcal{Y}$ of elements such that every element belongs to exactly one of these sets and AXY represents for every $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ the same sense. Then there exists within the elements of $\mathcal{X}$ or $\mathcal{Y}$ exactly one element L such that $AXLY$ is in natural order, independently of the choice of X and Y, as long as they do not coincide with L.

5. ***Notes and references***

For a more thorough discussion of the concepts mentioned in this introduction, including the axiomatics of projective geometry, see Coxeter [1955] [1965], Locher [1940] [1957], Cremona [1885], Forder [1927], Robinson [1940], Veblen/Young [1910], Veblen [1918].

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